A von Neumann–Morgenstern Representation Result without Weak Continuity Assumption

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In the paradigm of VON NEUMANN AND MORGENSTERN, a representation of affine preferences in terms of an expected utility can be obtained under the assumption of weak continuity. Since the weak topology is coarse, this requirement is a priori far from being negligible. In this work, we replace the assumption of weak continuity by monotonicity. More precisely, on the space of lotteries on an interval of the real line, it is shown that any affine preference order which is monotone with respect to the first stochastic order admits a representation in terms of an expected utility for some nondecreasing utility function. As a consequence, any affine preference order on the subset of lotteries with compact support, which is monotone with respect to the second stochastic order, can be represented in terms of an expected utility for some nondecreasing concave utility function. We also provide such representations for affine preference orders on the subset of those lotteries which fulfill some integrability conditions. The subtleties of the weak topology are illustrated by some examples.

Key Words: Affine Preference Order; von Neumann-Morgenstern Representation; Automatic Continuity; First Stochastic Order; Weak Continuity

Introduction

The problem of assessing uncertain monetary ventures by way of an expected utility dates back to the early stages of probability theory with the works of BERNOULLI and CRAMER on the St. Petersbourg paradox. However, in an approach consisting primarily in the study of preferences on lotteries, VON NEUMANN AND MORGENSTERN [15] provided in their seminal work the first formal systematic analysis of expected utility. These preferences are specified in terms of a *preference order*[1] $\succcurlyeq$ which is required to

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[1] A preference order is a transitive and complete binary relation $\succcurlyeq$. As usual, the notation $\succ$ and $\sim$ describe respectively the antisymmetric and equivalence relations.

satisfy the *independence* and *Archimedean* properties[2]. They show that such preference orders $\succcurlyeq$ admit an affine numerical representation U which is unique up to strictly monotone affine transformations. Preference orders admitting an affine numerical representation will be here referred to as *affine preference orders*. Under the additional condition that $\succcurlyeq$ is weakly continuous[3], the affine numerical representation U can be expressed in terms of an expected utility

$$U(\mu) = \int u \, d\mu, \tag{0.1}$$

for some continuous bounded utility function u, generically called *von Neumann–Morgenstern representation*.

The independence and Archimedean properties were disputed in several subsequent studies and extensions[4], but we here want to focus on this last requirement of weak continuity. In addition to being a strong mathematical requirement since the weak topology is coarse, it is actually truly questionable from a normative viewpoint; in contrast to the Archimedean property which is empirically as problematic to test but still has a certain normative appeal[5], the weak continuity condition is primarily a technical one. Starting with this observation as bottom line, the goal is to give another normatively satisfying condition which ensures the weak continuity of the preference order, and in turn, provides a representation of the form (0.1). The condition we consider is a monotonicity with respect to the first or second stochastic order, the normative and theoretical interest of which in the context of preferences has been studied by HADAR AND RUSSELL [9]. The works of NAMIOKA [14] and BORWEIN [3] include results when the monotonicity of convex functions on metrizable vector spaces induce automatically their continuity. These results were already mentioned and partly exploited in DRAPEAU AND KUPPER [5] in the context of general risk orders. However, these classical results on automatic continuity cannot be directly applied here. Indeed, the space of lotteries is a convex set and not a vector space. Moreover, the weak topology on the vector space of signed measures spanned by the lotteries is not metrizable.

To overcome these problems, we first show that any affine preference order which is monotone with respect to the first stochastic order is continuous for the variational norm. This does not imply the weak continuity of the preference order, mainly because lotteries cannot be approximated in the variational norm topology by simple lotteries[6]. There are actually examples of affine preference orders monotone with respect to the first stochastic order which are not weakly continuous. However, our second main result shows that these affine preference orders monotone with respect to the first stochastic order admit a von Neumann–Morgenstern representation in terms of a nondecreasing utility function. In this context, in spite of such a representation, the existence of a certainty equivalent is not necessarily ensured. Such a representation result does not hold for affine preference orders on arbitrary subsets of lotteries. However, we obtain similar representation results for the subspace of lotteries with compact support, showing in particular that monotonicity with respect to the second stochastic order is enough to ensure a von Neumann–Morgenstern representation. We also extend these results to some other canonical subspaces.

The structure of this paper is as follows. In the first section we present the setting and state the main representation results for affine preference orders on the set of lotteries. In the second section, we extend

[2] A preference order $\succcurlyeq$ satisfies the independence property if for any two lotteries μ, ν with $\mu \succ \nu$ holds $\alpha\mu + (1-\alpha)\eta \succ \alpha\nu + (1-\alpha)\eta$ for any lottery η and $\alpha \in]0,1[$. It satisfies the Archimedean property if for any three lotteries μ, ν, η with $\mu \succ \eta \succ \nu$ there exist $\alpha, \beta \in]0,1[$ such that $\alpha\mu + (1-\alpha)\nu \succ \eta \succ \beta\mu + (1-\beta)\nu$.

[3] A preference order is weakly continuous if the sets $\mathcal{L}(\mu) := \{\nu \mid \mu \succcurlyeq \nu\}$ and $\mathcal{U}(\mu) := \{\nu \mid \mu \preccurlyeq \nu\}$ are weakly closed for any lottery μ.

[4] For an overview of which, we refer to FISHBURN [6].

[5] In the sense that it is merely a one dimensional regularity assumption of the preferences along a segment between two alternatives.

[6] Convex combinations of Dirac measures.

these representation results for the particular subspaces of those lotteries with compact support as well as those which satisfy additional integrability conditions. In an appendix, we first give some standard definitions and discuss a Banach lattice structure for the first stochastic order. For the sake of readability, all the proofs are postponed to the Appendix.

1. Automatic Continuity and Main Representation Results

In the sequel, the affine preference order[7] $\succcurlyeq$ is defined on the set of lotteries $\mathcal{M}_1 := \mathcal{M}_1(\mathcal{S})$, that is, the set of probability distributions on the real line with support on a given interval $\mathcal{S} \subset \mathbb{R}$. As mentioned in the introduction, we do not require the weak $\sigma(\mathcal{M}_1, C_b)$-continuity[8] of the preference order to obtain a von Neumann–Morgenstern representation. We instead require the monotonicity with respect to the first stochastic order. The first stochastic order, denoted by $\trianglerighteq^1$, is defined by

$$\mu \trianglerighteq^1 \nu \quad \Longleftrightarrow \quad \int u\, d\mu \geq \int u\, d\nu \quad \text{for any nondecreasing } u \in C_b.$$

A preference order $\succcurlyeq$ is *monotone with respect to the first stochastic order* if $\mu \trianglerighteq^1 \nu$ implies $\mu \succcurlyeq \nu$.

We first present an automatic continuity result for the variational norm $\|\cdot\|$.

Proposition 1.1. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_1$ which is monotone with respect to the first stochastic order. Then, the preference order $\succcurlyeq$ is $\|\cdot\|$-continuous[9]. In particular, any affine numerical representation U of $\succcurlyeq$ is $\|\cdot\|$-continuous, and there exists $M > 0$ such that for any $\mu, \nu \in \mathcal{M}_1$,*

$$|U(\mu) - U(\nu)| \leq M \|\mu - \nu\|. \tag{1.1}$$

The proof in Appendix B.1 is based on ideas by NAMIOKA [14] and BORWEIN [3], whereby $\mathcal{M}_1$ is a convex set. On the other hand, an automatic $\sigma(\mathcal{M}_1, C_b)$-continuity proof along this line is not possible. Indeed, even though $\mathcal{M}_1$ is metrizable for the weak $\sigma(\mathcal{M}_1, C_b)$-topology, the vector space of signed measures denoted by ca, is no longer metrizable. For $\mathcal{S} = \mathbb{R}$, the preference order corresponding to the affine numerical representation

$$U(\mu) = \int 1_{[0,1]}(x) + 2 1_{]1,+\infty]}(x)\, \mu(dx), \tag{1.2}$$

is monotone with respect to the first stochastic order but is neither $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuous nor $\sigma(\mathcal{M}_1, C_b)$-lower semicontinuous. Notice that this example is however continuous for the finer topology $\sigma(\mathcal{M}_1, B_b)$ induced by the bounded functions of finite variation[10] $B_b := B_b(\mathcal{S})$. In fact, the previous automatic continuity result holds for this topology as stated in our second main result.

Theorem 1.2. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_1$ which is monotone with respect to the first stochastic order. Then, the preference order $\succcurlyeq$ is $\sigma(\mathcal{M}_1, B_b)$-continuous and it admits a von Neumann–Morgenstern representation*

$$U(\mu) = \int u\, d\mu, \quad \mu \in \mathcal{M}_1, \tag{1.3}$$

for some bounded nondecreasing utility function $u : \mathcal{S} \to \mathbb{R}$.

[7] Recall that a preference order is affine exactly when it satisfies the independence and Archimedean properties.

[8] Here, C_b denotes the vector space of continuous bounded functions $u : \mathcal{S} \to \mathbb{R}$. For the definition of the weak continuity, we refer to the Appendix A.

[9] Both $\mathcal{L}(\mu) := \{\eta \mid \mu \succcurlyeq \eta\}$ and $\mathcal{U}(\nu) = \{\eta \mid \eta \succcurlyeq \nu\}$ are $\|\cdot\|$-closed for any $\mu \in \mathcal{M}_1$.

[10] A function is of finite variation if it is the difference of two nondecreasing functions.

The proof of this theorem is treated in the Appendix C.1.

Remark 1.3. In the 1930s, KOLMOGOROV [12], NAGUMO [13], DE FINETTI [4] and HARDY, LITTLEWOOD, AND PÓLYA [10, Proposition 215, Page 158] studied the notion of means and obtained under the assumption of strict monotonicity with respect to the first stochastic order and the existence of a certainty equivalent[11] a representation of the form (1.3) in terms of a continuous increasing utility function u. Theorem 1.2 extends their representation results in the following sense. Firstly, $\mathcal{S}$ is not necessarily compact and we are not restricted to the set of simple lotteries. Secondly, we do not assume strict monotonicity and finally, the existence of certainty equivalent is not required. Note that a von Neumann–Morgenstern representation (1.3) does not necessarily imply the existence of a certainty equivalent[12]. For instance, the image of U in (1.2) is equal to the interval $[0, 2]$ whereas the image of its restriction to the set of Dirac measures is equal to the triple $\{0, 1, 2\}$. ♦

Remark 1.4. Monotonicity with respect to the first stochastic order is different from the *sure thing principle*[13] under which FISHBURN [7] provided a von Neumann–Morgenstern representation. For instance, the preference order in Example 2.1 is monotone with respect to the first stochastic order whereas the sure thing principle does not hold. ♦

Remark 1.5. FÖLLMER AND SCHIED [8, EXAMPLE 2.27] give an example of a non monotone affine preference order $\succcurlyeq$ which does not have a von Neumann–Morgenstern representation. ♦

Under a regularity condition on a subset of simple lotteries, we obtain the $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuity of the preference order.

Corollary 1.6. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_1$ fulfilling the same assumptions as in Theorem 1.2. Suppose furthermore that for any $t, t' \in \mathcal{S}$ and $\lambda \in [0, 1]$ the set*

$$\left\{ s \in \mathcal{S} \,\middle|\, \delta_s \succcurlyeq \lambda \delta_t + (1 - \lambda)\, \delta_{t'} \right\} \tag{1.4}$$

is closed in $\mathcal{S}$. Then, the preference order $\succcurlyeq$ is $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuous and it admits a von Neumann–Morgenstern representation

$$U(\mu) = \int u \, d\mu, \quad \mu \in \mathcal{M}_1, \tag{1.5}$$

for some bounded nondecreasing right-continuous utility function $u : \mathcal{S} \to \mathbb{R}$.

The proof of this corollary is treated in the Appendix C.2.

Remark 1.7. The assumption (1.4) only ensures the $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuity, in contrast to the standard automatic continuity results which guarantee the continuity. Indeed, for $\mathcal{S} = \mathbb{R}$, the preference order corresponding to the numerical representation

$$U(\mu) = \int 1_{[0,+\infty[}(x)\ \mu(dx),$$

is $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuous but not $\sigma(\mathcal{M}_1, C_b)$-continuous. This example again shows that the von Neumann–Morgenstern representation (1.5) does not necessarily have a certainty equivalent. ♦

[11] An element $\mu \in \mathcal{M}_1$ admits a certainty equivalent if there exists $c \in \mathcal{S}$ such that $\delta_c \sim \mu$.

[12] Note that the existence of a certainty equivalent is a strong requirement which even fails if u is strictly increasing with jumps.

[13] If μ is a lottery, A is a Borel set with $\mu(A) = 1$ and $s \in \mathbb{R}$, then $\delta_s \succcurlyeq \mu$ if $\delta_s \succcurlyeq \delta_t$ for all $t \in A$ and $\mu \succcurlyeq \delta_s$ if $\delta_t \succcurlyeq \delta_s$ for all $t \in A$.

2. Representation Results on Subsets of Lotteries

The boundedness of the utility function in the von Neumann–Morgenstern representation in Theorem 1.2 might sometimes be restrictive. In order to deal with unbounded[14] utility functions and monotonicity concepts such as the second stochastic order, we have to restrict the lotteries to subsets of $\mathcal{M}_1$. Note however that a von Neumann–Morgenstern representation might not exist on arbitrary subspaces as indicated by the following counter-example which is a modification of Example 2.26 in FÖLLMER AND SCHIED [8].

Example 2.1. As a subset of $\mathcal{M}_1\left(]0,+\infty[\right)$, consider

$$\mathcal{D} := \left\{ \mu \in \mathcal{M}_1(]0,+\infty[) \;\middle|\; \lim_{t\to+\infty} t^2\mu([t,+\infty[) \text{ exists and is finite} \right\}.$$

Define on $\mathcal{D}$ the function $U(\mu) := \lim_{t\to+\infty} t^2\mu([t,+\infty[)$ which is affine. Moreover, $\mu_1 \trianglerighteq^1 \mu_2$ implies that $\mu_1([t,+\infty[) \geq \mu_2([t,+\infty[)$ for all $t \in]0,+\infty[$ from which it follows that U is also monotone with respect to the first stochastic order. Nevertheless, U does not admit a von Neumann–Morgenstern representation since $U(\delta_x) = 0$ for any $x \in]0,+\infty[$. Furthermore, $U(\mu) = 0$ for all $\mu \in \mathcal{D}$ with compact support, which in view of Lemma C.1 implies that U is not continuous with respect to the variational norm $\|\cdot\|$. ◊

Considering the special subset $\mathcal{M}_{1,c} \subset \mathcal{M}_1$ of lotteries with compact support[15] we obtain a similar representation result as Theorem 1.2 in terms of not necessarily bounded utility functions. Here, we denote by $B := B(\mathcal{S})$ the functions $u : \mathcal{S} \to \mathbb{R}$ of finite variation which may be unbounded.

Proposition 2.2. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_{1,c}$ monotone with respect to the first stochastic order. Then, the preference order $\succcurlyeq$ is $\sigma(\mathcal{M}_{1,c}, B)$-continuous and it admits a von Neumann–Morgenstern representation*

$$U(\mu) = \int u\, d\mu, \quad \mu \in \mathcal{M}_{1,c}, \tag{2.1}$$

for a nondecreasing utility function $u : \mathcal{S} \to \mathbb{R}$.

Proof, Appendix C.3.

For lotteries with compact support we can also express the previous Proposition 2.2 under the assumption of monotonicity with respect to the second stochastic order $\trianglerighteq^2$ defined for any $\mu, \nu \in \mathcal{M}_{1,c}$ by

$$\mu \trianglerighteq^2 \nu \quad \Longleftrightarrow \quad \int u\, d\mu \geq \int u\, d\nu \quad \text{for any nondecreasing concave } u \in C,$$

where $C := C(\mathcal{S})$ is the set of continuous functions $u : \mathcal{S} \to \mathbb{R}$.

Corollary 2.3. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_{1,c}$ which is monotone with respect to the second stochastic order. Then $\succcurlyeq$ is $\sigma(\mathcal{M}_{1,c}, C)$-continuous and admits a von Neumann–Morgenstern representation*

$$U(\mu) = \int u\, d\mu, \quad \mu \in \mathcal{M}_{1,c}, \tag{2.2}$$

for a nondecreasing continuous concave utility function $u : \mathcal{S} \to \mathbb{R}$. In particular, any $\mu \in \mathcal{M}_{1,c}$ has a certainty equivalent, that is, there exists $c \in \mathcal{S}$ such that $\delta_c \sim \mu$.

[14] Indeed, boundedness is for instance not ensured for concave utility functions on the real line.

[15] A lottery $\mu \in \mathcal{M}_1$ has a compact support if there exists a compact $K \subset \mathcal{S}$ with $\mu(K) = 1$.

Proof Appendix C.4.

Finally, we study special subsets of $\mathcal{M}_1$ for which the ψ's moment exist. Following [8, Appendix A.6], we fix a so called gauge function, that is, a continuous function $\psi : S \to [1, +\infty[$ and define

$$\mathcal{M}_1^{\psi} := \mathcal{M}_1^{\psi}(\mathcal{S}) = \left\{ \mu \in \mathcal{M}_1 \,\middle|\, \int \psi \, d\mu < +\infty \right\}. \tag{2.3}$$

The set $\mathcal{M}_1^{\psi}$ spans the vector space ca^{ψ} of all signed measures $\mu \in ca$ for which the ψ's moment exists, that is, $\int \psi \, |d\mu| < +\infty$. By $B_{\psi} = B_{\psi}(\mathcal{S})$ we denote the vector space of all functions $u : \mathcal{S} \to \mathbb{R}$ with finite variation for which there exists a constant c such that $|u(x)| \leq c \cdot \psi(x)$ for all $x \in \mathcal{S}$. On this set we consider the $\sigma\left(\mathcal{M}_1^{\psi}, B_{\psi}\right)$-topology, that is the coarsest topology for which for all $f \in B_{\psi}$ the mapping

$$\mu \mapsto \int f \, d\mu$$

is continuous. For instance, for $\psi = \max(1, |x|)$, the set $\mathcal{M}_1^{\psi}(\mathbb{R})$ consists of all lotteries on $\mathbb{R}$ with finite first moment. Note that for $\psi \equiv M$ for some constant $M \in [1, +\infty[$, we have $\mathcal{M}_1^{\psi} = \mathcal{M}_1$ and $B_{\psi} = B_b$. On ca^{ψ} we also consider the ψ-variational norm given by

$$\|\mu\|_{\psi} := \|\psi \, d\mu\| = \int \psi \, |d\mu| , \quad \mu \in ca^{\psi},$$

where $\psi \, d\mu \in ca$ denotes the signed measure with Radon-Nikodym derivative[16] ψ with respect to μ. Proposition 1.1 translates in the present context as follows.

Proposition 2.4. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_1^{\psi}$ which is monotone with respect to the first stochastic order. Then, the preference order $\succcurlyeq$ is $\|\cdot\|_{\psi}$-continuous. In particular, any affine numerical representation U of $\succcurlyeq$ is $\|\cdot\|_{\psi}$-continuous, and there exists $M > 0$ such that*

$$|U(\mu) - U(\nu)| \leq M \, \|\mu - \nu\|_{\psi} , \tag{2.4}$$

for any $\mu, \nu \in \mathcal{M}_1^{\psi}$.

Proof, Appendix B.2.

On $\mathcal{M}_1^{\psi}$ we have the following representation result.

Theorem 2.5. *Let $\succcurlyeq$ be an affine preference order on $\mathcal{M}_1^{\psi}$ monotone with respect to the first stochastic order. Then, the preference order $\succcurlyeq$ is $\sigma(\mathcal{M}_1^{\psi}, B_{\psi})$-continuous and it admits a von Neumann–Morgenstern representation*

$$U(\mu) = \int u \, d\mu, \quad \mu \in \mathcal{M}_1^{\psi}, \tag{2.5}$$

for some nondecreasing utility function $u : \mathcal{S} \to \mathbb{R}$ satisfying $|u| \leq c \cdot \psi$ for some constant c.

Proof, Appendix C.5.

[16] For the Radon-Nykodym theorem, see [11, Theorem 2.10].

A. Notations

Throughout, $\mathcal{S}$ is a real interval. As mentioned before, $\mathcal{M}_1 := \mathcal{M}_1(\mathcal{S})$ and $ca := ca(\mathcal{S})$ are respectively the space of probability measures and signed measures on the Borel σ-algebra of $\mathcal{S}$. The space ca is the span of $\mathcal{M}_1$. The first stochastic order $\trianglerighteq^1$ on ca is defined as

$$\mu \trianglerighteq^1 \nu \quad \Longleftrightarrow \quad \int u\,d\mu \geq \int u\,d\nu \quad \text{for all nondecreasing } u \in C_b,$$

where C_b is the set of bounded continuous functions $u : \mathcal{S} \to \mathbb{R}$.

The variational norm $\|\cdot\|$ on ca is given by

$$\|\mu\| := \sup \left\{ \sum_n |\mu(A_n)| \;\middle|\; A_0, A_1, \ldots \text{ is a Borel partition of } \mathcal{S} \right\}, \quad \mu \in ca.$$

The weak topology $\sigma(\mathcal{M}_1, \mathcal{Y})$ on $\mathcal{M}_1$ is induced by the weak topology $\sigma(ca, \mathcal{Y})$ on ca, that is, the coarsest topology for which the linear functionals

$$\mu \mapsto \int u\,d\mu$$

are continuous for all $u \in \mathcal{Y}$, where either $\mathcal{Y} = B_b$, the set of bounded functions $u : \mathcal{S} \to \mathbb{R}$ with finite variation, or $\mathcal{Y} = C_b \subset B_b$.

B. Banach Lattice Structure for the First Stochastic Order

Given a convex set $\mathcal{X}$, an *affine function* $U : \mathcal{X} \to \mathbb{R}$ satisfies for all $x, y \in \mathcal{X}$ and $\lambda \in]0,1[$

$$U(\lambda x + (1-\lambda) y) = \lambda U(x) + (1-\lambda) U(y).$$

In the following $\mathcal{V}$ denotes the linear span of the set $\mathcal{X} - \mathcal{X}$. Note that $\mathcal{X}$ is not necessarily a subset of $\mathcal{V}$ unless $0 \in \mathcal{X}$. A similar argumentation as in the proof of Lemma B.1 below shows that

$$\mathcal{V} = \left\{ \lambda(\nu - \eta) \;\middle|\; \nu, \eta \in \mathcal{X} \text{ and } \lambda \geq 0 \right\}.$$

A preorder $\trianglerighteq$ on $\mathcal{X}$ is a *vector preorder* if there exists a convex cone[17] $\mathcal{K} \subset \mathcal{V}$ with $0 \in \mathcal{K}$ such that for any $\mu, \nu \in \mathcal{X}$,

$$\mu \trianglerighteq \nu \quad \Longleftrightarrow \quad \mu - \nu \in \mathcal{K}.$$

The same order is then also used on $\mathcal{V}$. A function $U : \mathcal{X} \to \mathbb{R}$ is *monotone* with respect to the vector preorder $\trianglerighteq$ if $U(\mu) \geq U(\nu)$ whenever $\mu \trianglerighteq \nu$.

Lemma B.1. *Given an affine function $U : \mathcal{X} \to \mathbb{R}$ monotone with respect to a vector preorder $\trianglerighteq$, there exists a linear function $\hat{U} : \mathcal{V} \to \mathbb{R}$ also monotone with respect to $\trianglerighteq$ such that for some $\mu_0 \in \mathcal{X}$*

$$\hat{U}(\mu - \mu_0) = U(\mu) - U(\mu_0), \quad \mu \in \mathcal{X}. \tag{B.1}$$

[17] A set $\mathcal{K} \subset \mathcal{V}$ is a convex cone if $\mathcal{K}$ is convex and $\lambda x \in \mathcal{K}$ for any $x \in \mathcal{K}$ and all $\lambda > 0$.

Proof. We define

$$\hat{U}(\mu) := \lambda(U(\nu) - U(\eta))$$

for $\mu := \lambda(\nu - \eta)$ with $\nu, \eta \in \mathcal{X}$ and $\lambda \geq 0$. It is well-defined since for $\mu := \lambda(\nu - \eta) = \lambda'(\nu' - \eta')$ with $\nu, \eta, \nu', \eta' \in \mathcal{X}$ and $\lambda, \lambda' > 0$

$$\frac{\lambda}{\lambda + \lambda'}\nu + \frac{\lambda'}{\lambda + \lambda'}\eta' = \frac{\lambda}{\lambda + \lambda'}\eta + \frac{\lambda'}{\lambda + \lambda'}\nu' \ \in \mathcal{X}$$

which, by means of the affinity of U, yields $\lambda(U(\nu) - U(\eta)) = \lambda'(U(\nu') - U(\eta'))$. Showing that $\hat{U}$ is linear and that (B.1) holds, follows analogously. It remains to show that $\hat{U}$ is monotone. Since $\hat{U}$ is linear and $\trianglerighteq$ is a vector preorder, it is enough to show that $\hat{U}(\mu) \geq 0$ for any $\mu \trianglerighteq 0$. Given such a $\mu = \lambda(\nu - \eta) \trianglerighteq 0$ for some $\nu, \eta \in \mathcal{X}$ and $\lambda \geq 0$, it follows that $\nu - \eta \in \mathcal{K}$, since $\mathcal{K}$ is a convex cone. From the monotonicity of U follows $U(\nu) \geq U(\eta)$ implying that $\hat{U}(\mu) \geq 0$. $\square$

Lemma B.2. *If $\mathcal{X} = \mathcal{M}_1$, then $\mathcal{V} := span(\mathcal{X} - \mathcal{X}) = \{\mu \in ca \,|\, \mu(\mathcal{S}) = 0\}$.*

Proof. The case $\subset$ is obvious. Conversely, for $\mu \in ca$ with $\mu(\mathcal{S}) = 0$, the Jordan-Hahn decomposition[18] yields $\mu = \mu^+ - \mu^-$ for $\mu^+, \mu^- \in ca_+$, where ca_+ denotes the subspace of nonnegative measures in ca. Without loss of generality, $\mu \neq 0$, and therefore $\lambda := \mu^+(\mathcal{S}) = \mu^-(\mathcal{S}) > 0$. Hence, $\mu = \lambda(\nu - \eta)$ where $\nu = \mu^+/\lambda \in \mathcal{M}_1$ and $\eta = \mu^-/\lambda \in \mathcal{M}_1$. $\square$

Lemma B.3. *If $\mathcal{X} = \mathcal{M}_1$, then the first stochastic order $\trianglerighteq^1$ is a vector preorder for*

$$\mathcal{K} = \left\{ \mu \in \mathcal{V} \,\Big|\, \int u d\mu \geq 0 \textit{ for any nondecreasing } u \in C_b \right\}$$

and $\mathcal{V} = \mathcal{K} - \mathcal{K}$, that is, for any $\mu \in \mathcal{V}$, there exist $\mu^1, \mu^2 \trianglerighteq^1 0$ such that $\mu = \mu^1 - \mu^2$.

Proof. That $\trianglerighteq^1$ is a vector preorder for the convex cone

$$\mathcal{K} = \left\{ \mu \in \mathcal{V} \,\Big|\, \int u d\mu \geq 0 \text{ for any nondecreasing } u \in C_b \right\}$$

is immediate from the definition of the first stochastic order. We are left to show that any $\mu \in \mathcal{V}$ can be decomposed in $\mu = \mu^1 - \mu^2$ where $\mu^1, \mu^2 \trianglerighteq^1 0$. Denote by $F(t) := \mu(]-\infty, t] \cap \mathcal{S})$ the cumulative distribution function of μ. From $\mu(\mathcal{S}) = 0$ follows $F(-\infty) = F(+\infty) = 0$. Since $t \mapsto F(t)$ has bounded variation and is right-continuous, the same holds for $F_1 := \max(F, 0)$ and $F_2 := \max(-F, 0)$. We can therefore define the signed measures $\mu^2 := -dF_1$ and $\mu^1 := -dF_2$, which by construction satisfy

$$\mu^1 - \mu^2 = dF_1 - dF_2 = dF = \mu$$

and $\mu^1(\mathcal{S}) = \mu^2(\mathcal{S}) = 0$ as $F_i(-\infty) = F_i(+\infty) = 0$ for $i = 1, 2$, showing that $\mu^1, \mu^2 \in \mathcal{V}$. Moreover, $\mu^i \trianglerighteq^1 0$ which follows by integration by parts,

$$\int u \, d\mu^i = -\int u \, dF_i = \int F_i \, du \geq 0, \quad \text{for any nondecreasing } u \in C_b,$$

for $i = 1, 2$ since du is a nonnegative measure. $\square$

[18] See for instance [11, Theorem 2.8].

Proposition B.4. *If $\mathcal{X} = \mathcal{M}_1$, then $(\mathcal{V}, \|\cdot\|)$ is a Banach lattice*[19] *for the first stochastic order.*

Proof. The fact that $\mathcal{V}$ is a lattice is immediate from the previous lemmata. It is also a Banach space, since ca is complete for the variational norm and $\mathcal{V} = \{\mu \in ca \,|\, \mu(\mathcal{S}) = 0\}$ is obviously closed in ca. Finally, for the decomposition of μ in μ^1 and μ^2 as in B.3, holds $\|\mu^1\| = \int |dF_2| \leq \int |dF| = \|\mu\|$ and $\|\mu^2\| = \int |dF_1| \leq \int |dF| = \|\mu\|$. Hence, from $\mu \trianglerighteq^1 \nu \trianglerighteq^1 0$ follows $\|\mu\| \geq \|\nu\|$. $\square$

Theorem B.5. *Given $\mathcal{X} = \mathcal{M}_1$ and some linear function $\hat{U} : \mathcal{V} \to \mathbb{R}$ monotone with respect to the first stochastic order, then $\hat{U}$ is continuous with respect to the variational norm $\|\cdot\|$.*

Proof. Since $(\mathcal{V}, \|\cdot\|)$ is a Banach space, the result follows from well-known automatic continuity results. For the sake of completeness, we present a proof in this case. Suppose by way of contradiction that $\hat{U}$ is not continuous. Then, there exists a sequence $\mu_k \in \mathcal{V}$ such that $\|\mu_k\| = 1$ and $\hat{U}(\mu_k) \geq 2^k$. According to Lemma B.3, this sequence decomposes in $\mu_k = \mu_k^1 - \mu_k^2$, where $\mu_k^i \trianglerighteq^1 0$ for $i = 1, 2$. From the Banach lattice structure of $\mathcal{V}$ for the first stochastic order, see Proposition B.4, it follows that $\mu := \sum_{k \geq 1} 2^{-k} \mu_k^1 \in \mathcal{V}$. Since $\hat{U}(\mu_k^1) \geq 0$ for all $k \in \mathbb{N}$, we end up with

$$\hat{U}(\mu) \geq \sum_{k=1}^{n} 2^{-k} \hat{U}\left(\mu_k^1\right) \geq \sum_{k=1}^{n} 2^{-k} \hat{U}(\mu_k) \geq n, \quad \text{for all } n,$$

in contradiction to the assumption that $\hat{U}(\mu) \in \mathbb{R}$. $\square$

B.1. Proof of Proposition 1.1

Proof. Let U be an affine numerical representation of the affine preference order $\succcurlyeq$ on $\mathcal{M}_1$. Due to Lemma B.1, there exists a monotone linear function $\hat{U} : \mathcal{V} \to \mathbb{R}$ and $\mu_0 \in \mathcal{M}_1$ such that

$$\hat{U}\left(\mu - \mu_0\right) = U\left(\mu\right) - U\left(\mu_0\right), \quad \mu \in \mathcal{M}_1,$$

where $\mathcal{V} = span\left(\mathcal{M}_1 - \mathcal{M}_1\right)$. Due to Theorem B.5, $\hat{U}$ is a continuous functional for the variational norm $\|\cdot\|$. Since $U(\mu) = \hat{U}(\mu - \mu_0) + U\left(\mu_0\right)$, it follows

$$\left|U\left(\mu\right) - U\left(\nu\right)\right| = \left|\hat{U}\left(\mu - \mu_0\right) - \hat{U}\left(\nu - \mu_0\right)\right| \leq \|\hat{U}\|_* \left\|\mu - \nu\right\|, \quad \text{for any } \mu, \nu \in \mathcal{M}_1,$$

for the operator norm $\|\hat{U}\|_* := \sup\{\hat{U}\left(\mu\right) \mid \mu \in ca\,,\ \|\mu\| \leq 1\}$ and the proof is complete. $\square$

B.2. Proof of Proposition 2.4

Proof. Using the fact that $\mu \in ca^\psi$ is equivalent to $\psi d\mu \in ca$ where $\psi d\mu$ is the measure in ca with Radon-Nikodym derivative with respect to μ equal to ψ, a similar argumentation as in the proof of Lemma B.2 and B.3 shows that they also hold in the case where $\mathcal{X} = \mathcal{M}_1^\psi$ and $\mathcal{K}$ is replaced by

$$\mathcal{K} := \left\{ \mu \in \mathcal{M}_1^\psi \;\middle|\; \int u d\mu \geq 0 \text{ for any nondecreasing } u \in C_b \right\}$$

and ca by ca^ψ. As for Proposition B.4, again, direct inspection shows that $\mathcal{V}^\psi := span\left(\mathcal{M}_1^\psi - \mathcal{M}_1^\psi\right)$ is complete for the norm $\|\cdot\|_\psi$. Hence, Theorem B.5 also holds in the case where $\mathcal{X} = \mathcal{M}_1^\psi$ with the norm $\|\cdot\|_\psi$. The proof of Proposition 2.4 is then analogous to the proof of Proposition 1.1. $\square$

[19] For the definition of a Banach lattice we refer to Chapter 9 in [1].

C. Technical Proofs

C.1. Proof of Theorem 1.2

Before addressing the proof of Theorem 1.2, we need the following lemma.

Lemma C.1. *The space of lotteries with compact support $\mathcal{M}_{1,c}$ is $\|\cdot\|$-dense in $\mathcal{M}_1$.*

Proof. Consider an increasing sequence K_n of compacts in $\mathcal{S}$ such that $\cup_n K_n = \mathcal{S}$. Take $\mu \in \mathcal{M}_1$ and for K_n big enough such that $\mu(K_n) > 0$. Define the lottery $\mu_n \in \mathcal{M}_{1,c}$ by

$$\mu_n(A) := \mu(A \cap K_n) / \mu(K_n), \quad \text{for any Borel set } A \subset \mathcal{S}.$$

We further define the measures $\tilde{\mu}_n$ and $\bar{\mu}_n$ by $\tilde{\mu}_n(A) := \mu(A \cap K_n^c)$ and $\bar{\mu}_n(A) := \mu(A \cap K_n)$. Then

$$\|\mu - \mu_n\| \leq \|\bar{\mu}_n - \mu_n\| + \|\tilde{\mu}_n\| \leq \left|1 - \frac{1}{\mu(K_n)}\right| \|\mu\| + \mu(K_n^c) \xrightarrow[n \to +\infty]{} 0$$

due to the σ-additivity of μ. □

Proof (of Theorem 1.2). For any $x \in \mathcal{S}$, we define $u(x) := U(\delta_x)$ which is a bounded function. Indeed, according to Proposition 1.1, there exists $M > 0$ such that

$$|u(x) - u(y)| = |U(\delta_x) - U(\delta_y)| \leq M \|\delta_x - \delta_y\| \leq 2M, \quad \text{for all } x, y \in \mathcal{S}.$$

By monotonicity with respect to the first stochastic order, u is moreover a nondecreasing function, and therefore it has only a countable number of discontinuities.

Step 1. In this step, we suppose that $\mathcal{S} = [a, b]$ for $a < b$ and that u is right-continuous in a and left-continuous in b. We show that for any $\mu \in \mathcal{M}_1([a, b])$

$$U(\mu) = \int u \, d\mu. \tag{C.1}$$

We denote $u(x+) := \inf\{u(y) \mid y > x\}$, and $u(x-) := \sup\{u(y) \mid y < x\}$, and by u_c the continuous part of u on $[a, b]$ defined as

$$u_c(x) := u(x) - \sum_{a \leq y < x} \big[u(y+) - u(y)\big] - \sum_{a < y \leq x} \big[u(y) - u(y-)\big].$$

From the boundedness of u, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{\substack{y \in]a,b[\\ |u(y+) - u(y-)| < \delta}} \big[u(y+) - u(y-)\big] \leq \varepsilon. \tag{C.2}$$

Fix some $\varepsilon > 0$, and according to relation C.2, denote by $s_1, \ldots, s_{N_\varepsilon}$ the finite number of jump points of u of size greater than δ, that is, the points[20] $\{x \in]a, b[\mid u(x+) - u(x-) \geq \delta\}$. We then define

$$\begin{aligned} u_\varepsilon(x) :=& u_c(x) + \Delta^\varepsilon(x) \\ :=& u_c(x) + \sum_{k=1}^{N_\varepsilon} \Big[u(s_k+) - u(s_k)\Big] 1_{]s_k, b]}(x) + \Big[u(s_k) - u(s_k-)\Big] 1_{[s_k, b]}(x). \end{aligned}$$

[20] Note that a jump point here is a positive difference between the left and right-continuous version of u.

By definition of u_ε we have $\|u - u_\varepsilon\|_\infty < \varepsilon$ on $[a, b]$.

We now consider some subdivision $\sigma_n = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$ such that $|\sigma_n| \leq 1/n$, where $|\sigma_n|$ denotes the mesh of the subdivision. We also suppose that this subdivision is fine enough that each jump point of u of size greater than δ on $[a, b]$ is also part of this subdivision. Let $i_1, i_2, \ldots, i_{N_\varepsilon}$ denote the ordered indices of the subdivision such that $x_{i_k} = s_k$.

Given $\mu \in \mathcal{M}_1([a, b])$, denote by $\beta_k = \mu(\{s_k\})$. If $\sum_{k=1}^{N_\varepsilon} \beta_k = 1$, then $\mu = \sum_{k=1}^{N_\varepsilon} \beta_k \delta_{s_k}$ and (C.1) trivially holds. If $\sum_{k=1}^{N_\varepsilon} \beta_k < 1$, consider μ_c given by

$$\mu = \left(1 - \sum_{k=1}^{N_\varepsilon} \beta_k\right) \mu_c + \sum_{k=1}^{N_\varepsilon} \beta_k \delta_{s_k},$$

which is a probability measure such that $\mu_c(\{s_k\}) = 0$ and therefore

$$\lim_{h \searrow 0} \mu_c([s_k, s_k + h[) = 0, \quad k = 1, \ldots, N_\varepsilon. \tag{C.3}$$

We now define

$$\hat{\mu}_n = \sum_{r=0}^{n-1} \alpha_r \delta_{x_r}, \quad \text{and} \quad \check{\mu}_n = \sum_{r=0}^{n-1} \alpha_r \delta_{x_{r+1}},$$

Where $\alpha_r = \mu_c([x_r, x_{r+1}[)$ if $r < n-2$ and $\alpha_{n-1} = \mu_c([x_{n-1}, b])$. By definition, it is obvious that $\check{\mu}_n \trianglerighteq^1 \mu_c \trianglerighteq^1 \hat{\mu}_n$ and therefore

$$\int u \, d\check{\mu}_n \geq \int u \, d\mu_c \geq \int u \, d\hat{\mu}_n. \tag{C.4}$$

From the monotonicity of U with respect to the first stochastic order also holds

$$\int u \, d\check{\mu}_n = U(\check{\mu}_n) \geq U(\mu_c) \geq U(\hat{\mu}_n) = \int u \, d\hat{\mu}_n. \tag{C.5}$$

Subtracting (C.4) from (C.5) yields

$$\left| U(\mu_c) - \int u \, d\mu_c \right| \leq \left| \int u \, d\check{\mu}_n - \int u \, d\hat{\mu}_n \right| = \int u \, d\check{\mu}_n - \int u \, d\hat{\mu}_n. \tag{C.6}$$

Together with the affinity of U and the definition of u, it follows that

$$\begin{aligned}
\left| U(\mu) - \int u \, d\mu \right| &= \left(1 - \sum_{k=0}^{N_\varepsilon} \beta_k\right) \left| U(\mu_c) - \int u \, d\mu_c \right| \\
&\leq \int u \, d\check{\mu}_n - \int u \, d\hat{\mu}_n = \int (u - u_\varepsilon) \, d(\check{\mu}_n - \hat{\mu}_n) + \int u_\varepsilon \, d(\check{\mu}_n - \hat{\mu}_n) \\
&\leq 2\varepsilon + \int u_c \, d(\check{\mu}_n - \hat{\mu}_n) + \int \Delta^\varepsilon \, d(\check{\mu}_n - \hat{\mu}_n).
\end{aligned} \tag{C.7}$$

We finally show that that $\int u_c \, d\left(\check{\mu}_n - \hat{\mu}_n\right) + \int \Delta^\varepsilon \, d\left(\check{\mu}_n - \hat{\mu}_n\right)$ can be arbitrarily small as the mesh of the subdivision converges to 0 for n going to ∞. As for the first term to estimate, since u_c is uniformly continuous on the compact $[a, b]$, it follows that

$$0 \leq \int u_c \, d\left(\check{\mu}_n - \hat{\mu}_n\right) = \sum_{r=0}^{n-1} \alpha_r \left[u_c\left(x_{r+1}\right) - u_c\left(x_r\right)\right] \leq \sup_{r=0,\ldots,n-1} \left|u_c\left(x_{r+1}\right) - u_c\left(x_r\right)\right| \xrightarrow[n \to +\infty]{} 0.$$

By definition, the other term to estimate yields

$$\begin{aligned} 0 \leq \int \Delta^\varepsilon \, d\left(\check{\mu}_n - \hat{\mu}_n\right) = & \sum_{r=0}^{n-1} \sum_{k=1}^{N_\varepsilon} \alpha_r \left[u\left(s_k+\right) - u\left(s_k\right)\right] \left[1_{]s_k, b]}\left(x_{r+1}\right) - 1_{]s_k, b]}\left(x_r\right)\right] \\ & + \sum_{r=0}^{n-1} \sum_{k=1}^{N_\varepsilon} \alpha_r \left[u\left(s_k\right) - u\left(s_k-\right)\right] \left[1_{[s_k, b]}\left(x_{r+1}\right) - 1_{[s_k, b]}\left(x_r\right)\right]. \end{aligned}$$

However, the terms $1_{]s_k, b]}\left(x_{r+1}\right) - 1_{]s_k, b]}\left(x_r\right)$ and $1_{[s_k, b]}\left(x_{r+1}\right) - 1_{[s_k, b]}\left(x_r\right)$ are equal to 0 for any $r = 0, \ldots, n$ except for $r = i_k$ where $x_{i_k} = s_k$ in which case $1_{]s_k, b]}\left(x_{r+1}\right) - 1_{]s_k, b]}\left(x_r\right) = 1$ and for $r = i_k - 1$ where $x_{i_k+1} = s_k$ in which case $1_{[s_k, b]}\left(x_{r+1}\right) - 1_{[s_k, b]}\left(x_r\right) = 1$. Since N_ε does not depend on n, it follows

$$\begin{aligned} 0 \leq \int \Delta_l^\varepsilon \, d\left(\check{\mu}_n - \hat{\mu}_n\right) &= \sum_{k=1}^{N_\varepsilon} \Big(u\left(s_k+\right) - u\left(s_k\right)\Big) \alpha_{i_k} + \Big(u\left(s_k\right) - u\left(s_k-\right)\Big) \alpha_{i_k-1} \\ &\leq 2 N_\varepsilon \left\|u\right\|_\infty \sup_{k=1,\ldots,N_\varepsilon} \Big[\mu_c\left(\left[s_k, x_{i_k+1}\right[\right) + \mu_c\left(\left[x_{i_k-1}, s_k\right[\right)\Big] \\ &\leq 2 N_\varepsilon \left\|u\right\|_\infty \sup_{k=1,\ldots,N_\varepsilon} \left[\mu_c\left(\left[s_k, s_k + \frac{1}{n}\right[\right) + \mu_c\left(\left[s_k - \frac{1}{n}, s_k\right[\right)\right]. \end{aligned}$$

By means of relation (C.3), holds

$$\sup_{k=1,\ldots,N_\varepsilon} \mu_c\left(\left[s_k, s_k + \frac{1}{n}\right[\right) \xrightarrow[n \to +\infty]{} 0,$$

and since $\left[s_k - \frac{1}{n}, s_k\right[\xrightarrow[n \to +\infty]{} \emptyset$, the σ-additivity of μ_c yields also

$$\sup_{k=1,\ldots,N_\varepsilon} \mu_c\left(\left[s_k - \frac{1}{n}, s_k\right[\right) \xrightarrow[n \to +\infty]{} 0.$$

Hence, for any $\varepsilon > 0$ and sufficiently large n holds

$$\left|U\left(\mu\right) - \int u d\mu\right| \leq 3\varepsilon$$

showing (C.1).

Step 2. In this step, suppose now that $\mathcal{S}$ is an open interval. Since u has countably many jump points, we can find an increasing sequence $\left[a_m, b_m\right]$ of compact intervalls such that $\mathcal{S} = \cup_n \left[a_n, b_n\right]$ where u is continuous in each a_n and b_n. Consider now some $\mu \in \mathcal{M}_1$ and let us show that

$$U\left(\mu\right) = \int u \, d\mu.$$

To this aim, fix some $\varepsilon > 0$. Since U is $\|\cdot\|$-continuous by Proposition 1.1, it follows by means of Lemma C.1 that there exist some $\mu_\varepsilon \in \mathcal{M}_{1,c}$ such that

$$|U(\mu) - U(\mu_\varepsilon)| \leq \varepsilon/2,$$

and

$$\left| \int u\, d\mu - \int u\, d\mu_\varepsilon \right| \leq \varepsilon/2.$$

Fix $m_0 \in \mathbb{N}$ such that $\mu_\varepsilon \in \mathcal{M}_1([a_{m_0}, b_{m_0}])$. Due to relation (C.1) it follows

$$\left| U(\mu) - \int u\, d\mu \right| \leq |U(\mu) - U(\mu_\varepsilon)| + \left| U(\mu_\varepsilon) - \int u\, d\mu_\varepsilon \right| + \left| \int u\, d\mu_\varepsilon - \int u\, d\mu \right| \leq \varepsilon.$$

Step 3. We are left to show that the von Neumann-Morgenstern representation also holds in the case where $\mathcal{S}$ is not open, for instance of the form $[a, b[$ for $a, b \in \mathbb{R}$. The cases $[a, b]$, $[a, +\infty[$, $]a, b]$ and $]-\infty, b[$ work analogously. For any $\mu \in \mathcal{M}_1$, there exist $\lambda \in [0, 1]$ and $\nu \in \mathcal{M}_1(]a, b[)$ such that $\mu = \lambda\delta_a + (1 - \lambda)\nu$. By affinity it follows from the previous representation result that

$$U(\mu) = \lambda u(a) + (1 - \lambda) \int_{]a,b[} u\, d\nu = \int_{[a,b[} u\, d\mu,$$

and this ends the proof. □

C.2. Proof of Corollary 1.6

Proof. Let U be an affine numerical representation of $\succcurlyeq$. Due to Theorem 1.2, U has a von Neumann–Morgenstern representation

$$U(\mu) = \int u\, d\mu, \quad \mu \in \mathcal{M}_1,$$

for some bounded nondecreasing utility function $u : \mathcal{S} \to \mathbb{R}$. Let us show that u is right-continuous. Suppose by way of contradiction that u is not right-continuous in $t \in \mathcal{S}$. Then, there exists $m \in \mathbb{R}$ such that $u(t) < m < \inf_{s>t} u(s)$. Pick now $t' > t$ and $\lambda \in [0, 1]$ such that $m = \lambda u(t) + (1 - \lambda) u(t')$. Hence

$$\left\{ s \in \mathcal{S} \,\middle|\, u(s) \geq m \right\} = \{ s \in \mathcal{S} \,|\, \delta_s \succcurlyeq \lambda\delta_t + (1 - \lambda)\delta_{t'} \} =]t, +\infty[\cap \mathcal{S}$$

which is relatively open in $\mathcal{S}$ in contradiction to (1.4). Thus, u is right-continuous. Since any bounded nondecreasing right-continuous function is a pointwise limit from above of bounded continuous functions, the Lebesgue monotone convergence theorem implies that U is the limit of a decreasing sequence of $\sigma(\mathcal{M}_1, C_b)$-continuous affine functions, hence is $\sigma(\mathcal{M}_1, C_b)$-upper semicontinuous. □

C.3. Proof of Proposition 2.2

Proof. In case that $\delta_{x_1} \sim \delta_{x_2}$ for all $x_1, x_2 \in \mathcal{S}$, the result is obvious. Otherwise, pick $c_2 > c_1$ with $\delta_{c_2} \succ \delta_{c_1}$. Consider a countable increasing sequence of closed intervals $[a_n, b_n] \subset \mathcal{S}$ such that $c_1, c_2 \in [a_0, b_0]$ and $\bigcup_n [a_n, b_n] = \mathcal{S}$, implying that $\mathcal{M}_{1,c} = \cup_n \mathcal{M}_1([a_n, b_n])$. The restriction of $\succcurlyeq$

to $\mathcal{M}_1([a_n, b_n])$ fulfils the conditions of Theorem 1.2, hence, this restriction admits a von Neumann–Morgenstern representation of the form

$$U_n(\mu) = \int_{[a_n,b_n]} u_n \, d\mu, \quad \mu \in \mathcal{M}_1([a_n, b_n]),$$

for some bounded nondecreasing utility functions $u_n : [a_n, b_n] \to \mathbb{R}$. Let us now show that we can construct a von Neumann–Morgenstern representation of $\succcurlyeq$. Since any affine numerical representation is unique up to strict affine transformation, the function u_n is uniquely determined if we fix $u_n(c_1) = 0$ and $u_n(c_2) = 1$ for all $n \in \mathbb{N}$. This implies in particular that $u_n = u_m$ on $[a_m, b_m]$ for any $n > m$. We then obtain a nondecreasing, not necessarily bounded, utility function $u : \mathcal{S} \to \mathbb{R}$. Moreover, for any $\mu \in \mathcal{M}_{1,c}$, we have

$$U(\mu) = \int u \, d\mu = \int_{[a_n,b_n]} u_n \, d\mu = U_n(\mu),$$

for any n such that $\mu \in \mathcal{M}_1([a_n, b_n])$, and this ends the proof. □

C.4. Proof of Corollary 2.3

Proof. If a preference order $\succcurlyeq$ is monotone with respect to the second stochastic order, then it is monotone with respect to the first stochastic order. Indeed, suppose that $\mu \trianglerighteq^1 \nu$, then by definition holds $\mu \trianglerighteq^2 \nu$ since any continuous concave nondecreasing function is a fortiori continuous nondecreasing and therefore, $\mu \succcurlyeq \nu$. Let U be an affine numerical representation for $\succcurlyeq$ which by means of Proposition 2.2 has a von Neumann–Morgenstern representation of the form

$$U(\mu) = \int u \, d\mu, \quad \mu \in \mathcal{M}_{1,c},$$

for some continuous increasing utility function u. This utility function is concave since for any $x, y \in \mathcal{S}$ and $\lambda \in [0, 1]$, the lottery $\delta_{\lambda x + (1-\lambda)y}$ dominates in the second stochastic order the convex combination $\lambda\delta_x + (1-\lambda)\delta_y$ and therefore

$$u(\lambda x + (1-\lambda)y) = U\left(\delta_{\lambda x+(1-\lambda)y}\right) \geq U(\lambda\delta_x + (1-\lambda)\delta_y) = \lambda u(x) + (1-\lambda) u(y),$$

which ends the proof. □

C.5. Proof of Theorem 2.5

Proof. Let U be an affine numerical representation of $\succcurlyeq$ on $\mathcal{M}_1^{\psi}$. Define $u(x) = U(\delta_x)$ for $x \in \mathcal{S}$. Due to Proposition 2.4, U is $\|\cdot\|_{\psi}$-continuous, hence

$$|u(x)| \leq M\left(\|\delta_x\|_{\psi} + \|\delta_y\|_{\psi}\right) + |u(y)| = M\big(\|\psi d\delta_x\| + \|\psi d\delta_y\|\big) + |u(y)| \leq \tilde{M}\psi(x), \quad x \in \mathcal{S},$$

for some $y \in \mathcal{S}$ and constants $M, \tilde{M} > 0$ where the last inequality holds since $\psi \geq 1$. Hence, $u \in B_{\psi}$. Let us show now that $U(\mu) = \int u \, d\mu$ for any $\mu \in \mathcal{M}_1^{\psi}$. Adapting the proof of Lemma C.1, $\mathcal{M}_{1,c}$ is

$\|\cdot\|_\psi$-dense in $\mathcal{M}_1^\psi$. Moreover, for any compact $K \subset \mathcal{S}$, the norms $\|\cdot\|$ and $\|\cdot\|_\psi$ are obviously equivalent on $\mathcal{M}_1(K) \subset \mathcal{M}_1^\psi$. We can therefore apply Proposition 2.2 to get

$$U(\mu) = \int_K u \, d\mu,$$

for any $\mu \in \mathcal{M}_{1,c}(K)$. A similar argumentation as in the second step in the proof of Theorem 1.2 ends the proof. $\square$

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