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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 4

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtrated probability space where $\mathbf{T} = \mathbb{N}_0$.

Exercise 4.1. *(10 points)*

(a) Let X and Y be two super-martingales. Show that $X \wedge Y$ is a super-martingale;

(b) Let X be a predictable martingale, show that $X_t = X_0$ for every t.

(c) Let $\xi \in L^1$, show that $X_t := E[\xi|\mathcal{F}_t]$ defines a martingale X;

(d) Show, or find a counter example for the following assertion: Let X be an adapted process such that X_t is integrable and $E[X_0] = E[X_t]$ for every t. Then X is a martingale.

Exercise 4.2. *(5 points) Let $I \subseteq \mathbf{T}$ be such that $\sup I = \infty$. Show that every super-martingale X such that $E[X_0] \leq E[X_t]$ for every $t \in I$ is a martingale.*

Exercise 4.3. *(10 points) Let X be an adapted process such that $E[\sup_t |X_t|] < \infty$. Denote by $\mathcal{T}$ the set of all stopping times and let $T \in \mathbb{N}_0$ be an arbitrary time horizon. We define recursively*

$$S_T = X_T \quad \text{and} \quad S_t = \max\left\{E\left[S_{t+1}|\mathcal{F}_t\right]; X_t\right\}, \quad t \leq T-1.$$

Define further $\tau^t = \inf\{s \colon s \geq t \text{ and } S_s = X_s\}$ for every $t = 0, \ldots, T$.

(a) Show that $S = (S_t)_{t=0,\ldots,T}$ is a super-martingale such that $S_t \geq X_t$ for every $t = 0, \ldots, T$;

(b) Let $U = (U_t)_{t=0,\ldots,T}$ be a super-martingale such that $U_t \geq X_t$ for every $t = 0, \ldots, T$. Show that $U_t \geq S_t$ for every $t = 0, \ldots, T$;

(c) Show that $E[X_{\tau^t}|\mathcal{F}_s] = E[S_t|\mathcal{F}_s]$ for every $s \leq t \leq T$ and conclude that

$$E\left[X_{\tau^t}\right] = E\left[S_t\right] = \max_{\{\tau \in \mathcal{T} \colon t \leq \tau \leq T\}} E\left[X_\tau\right]$$

(d) We denote by $S^T = (S^T_t)_{t=0,\ldots,T}$ the process defined in (a) whereby, we stress the dependence on the time horizon due to its recursive definition. Clearly, $S_t = \lim_{T\to\infty} S^T_t$ defines a process S. Show that S is a super-martingale for which holds $S \geq X$. Show further that for every other super-martingale U such that $U \geq X$ it follows $U \geq S$.

Exercise 4.4. *(Bonus 10 points) Consider now our example of coin tossing but infinitely many times. As seen, the state space is defined as follows*

$$\Omega = \prod_{t\in\mathbb{N}} \{-1, 1\} = \{-1, 1\}^{\mathbb{N}} = \{\omega = (\omega_t) \colon \omega_t = \pm 1 \text{ for every } t\}$$

On each $\Omega_t = \{-1, 1\}$ we consider the σ-algebra $\mathcal{F}_t = \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\}$ and on Ω the product σ-algebra $\mathcal{F} = \otimes \mathcal{F}_t$.

(a) Show that the collection $\mathcal{R}$ of finite product cylinders

$$A = \{\omega = (\omega_t) \in \Omega \colon \omega_{t_k} = e_k, k = 1, \dots n\} \tag{4.1}$$

for a given set of values $e_k \in \{-1, 1\}$, and times $t_k \in \mathbb{N}$, $k = 1, \dots, n$, together with the empty-set is a semi-ring.

(b) For $p \in [0, 1]$, we define $P : \mathcal{R} \to [0, 1]$ as follows

$$P[\emptyset] = 0 \quad \text{and} \quad P[A] = p^l (1-p)^{n-l}$$

for every $A \in \mathcal{R}$ of the form (4.1) where l is equal to the number of those $k = 1, \dots, n$ where $e_k = 1$. Show that P defines a content.

(c) Show that for every finite family $(A_k)_{k \leq n}$ of elements in $\mathcal{R}$ and $A \in \mathcal{R}$ such that $A \subseteq \cup_{k \leq n} A_k$, it holds[1]

$$P[A] \leq \sum_{k \leq n} P[A_k].$$

(d) Show that for every countable family (A_n) of elements in $\mathcal{R}$ and $A \in \mathcal{R}$ such that $A \subseteq \cup A_n$, it holds[2]

$$P[A] \leq \sum P[A_n],$$

and deduce using Caratheodory's theorem that P extends uniquely to a probability measure P on $\mathcal{F}$.

(e) Defining the process X by $X_t(\omega) = \omega_t$, describe the filtration generated by X.

Due date: Upload before Monday 2015.10.26 14:00.

[1] Consider first the case where $n = 2$ and A_1, A_2 are one dimensional product cylinder.

[2] Show that there exists a finite n_0 such that $A \subseteq \cup_{k \leq n_0} A_k$.