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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 8

Exercise 8.1. *(8 points)*

Recall from the lecture that a family $(\mathcal{C}^i)$ *of collections of elements in* $\mathcal{F}$ *on some probability space* $(\Omega, \mathcal{F}, P)$ *is called* independent *if for every finite collection* $(A_{i_k})_{k \leq n}$ *with* $A_{i_k} \in \mathcal{C}^{i_k}$ *for every* $k = 1, \ldots, n$*, it holds*

$$P\left[A_{i_1} \cap \cdots \cap A_{i_n}\right] = \prod_{k \leq n} P\left[A_{i_k}\right]$$

And random variables of a family (X^i) *are independent if* $(\sigma(X_i))$ *is an independent family.*

- *Such a family* $(\mathcal{C}^i)$ *is called* pairwise independent *if* $P[A_{i_1} \cap A_{i_2}] = P[A_{i_1}]P[A_{i_2}]$ *for every* $A_{i_1} \in \mathcal{C}^{i_1}$, $A_{i_2} \in \mathcal{C}^{i_2}$ *and every* i_1, i_2*. Pairwise independence is therefore a weaker statement than independence. It is actually strictly weaker.*

  Find three events A, B *and* C *in some probability space that are pairwise independent but not independent.*

- *Show that two random variables* X, Y *are independent, if and only it*

  $$P_{(X,Y)} = P_X \otimes P_Y$$

  where

  $$P_{(X,Y)}[A] := P[(X,Y) \in A], \quad P_X[A_1] := P[X \in A_1] \quad \text{and} \quad P_Y[A_2] := P[Y \in A_2]$$

  for Borel sets $A_1, A_2 \in \mathcal{B}(\mathbb{R})$ *and* $A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Exercise 8.2. *(8 points)*

Let $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\mathcal{G}$ *be a* σ*-algebra such that* $\mathcal{G} \subseteq \mathcal{F}$*. Let further* (A_n) *be a sequence of pairwise disjoint elements of* $\mathcal{F}$ *such that* $P[A_n] > 0$ *for every* n*. Define* $\mathcal{G} = \sigma(A_n \colon n)$ *the* σ*-algebra generated by the sequence* (A_n)*. Show that*

(i) for every $B \in \mathcal{F}$ *it holds*

$$P\left[B|\mathcal{G}\right] := E\left[1_B|\mathcal{G}\right] = \sum P\left[B|A_n\right] 1_{A_n}$$

where $P[B|A_n] := P[B|\sigma(A_n)] = P[B \cap A_n]/P[A_n]$.

(ii) for every $X \in L^1$*, it holds*

$$E\left[X|\mathcal{G}\right] = \sum \frac{E\left[1_{A_n} X\right]}{P[A_n]} 1_{A_n}$$

Exercise 8.3. *(8 points)*

Let $(\Omega, \mathcal{F}, P)$ *be a probability space and* $\mathcal{G}$ *be a* σ*-algebra such that* $\mathcal{G} \subseteq \mathcal{F}$*. Let also* $X \in L^1$ *be a random variable such that* $X > 0$ *almost surely and* $E[X] = 1$.

(i) Show that $Q : \mathcal{F} \to [0,1]$, defined as $Q[A] = E[1_A X]$ defines a probability measure which is equivalent to P[1] *and for which holds*

$$E_Q[Y] = E[XY]$$

for every positive random variable Y,

(ii) Show that for every $\mathcal{G}$-measurable positive random variable Y, it holds

$$E_Q[Y] = E[E[X|\mathcal{G}]Y]$$

(iii) Let $\mathcal{H}$ be a σ-algebra such that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and Y be a $\mathcal{G}$-measurable positive random variable. Show that it holds

$$E_Q[Y|\mathcal{H}] = \frac{1}{E[X|\mathcal{H}]} E[XY|\mathcal{H}]$$

(iv) Let $\mathbb{F} = (\mathcal{F})_{0 \leq t \leq T}$ be a filtration where T is a finite integer. Define the process X by $X_t = E[X|\mathcal{F}_t]$ for every $t = 1, \ldots, T$. Show that a process Y is a martingale with respect to the measure Q if and only if the process $XY = (X_t Y_t)_{1 \leq t \leq T}$ is a martingale with respect to the measure P.

Exercise 8.4. *(10 points)*
Be careful, in this exercise, we make everything upside down.

- *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{0 \leq t}$ be a sequence of σ-algebras such that $\mathcal{F} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_t \supseteq \ldots$. Suppose that X is a backward-martingale, that is such that*
  - *X_t is $\mathcal{F}_t$-measurable for every t;*
  - *X_t is integrable for every t;*
  - *$E[X_t|\mathcal{F}_{t+1}] = X_{t+1}$.*

  Inspired by Doob's upcrossing's inequality, show that $X_t \to X_T$ almost surely where $X_T \in L^1$.

- *Let (X_t) be a sequence of independent, identically distributed and integrable random variable. Show using the previous point that $(\sum_{s \leq t} X_s)/t$ converges almost surely.*[2]

Suppose now that

Due date: Upload before Monday 2015.23.09 14:00.

[1] That is $Q[A] = 0$ if and only if $P[A] = 0$.
[2] Hint: Show that $E[X_1|\sigma(\sum_{s \leq t} X_s, X_{t+1}, \ldots)] = \sum_{s \leq t} X_s$.