"STOCHASTIC PROCESSES" – HOMEWORK SHEET 9

Exercise 9.1. *(5 points) Let X and Y be two integrable random variables independent of each other and identically distributed. Show that*

$$E\left[X|X+Y\right] = \frac{X+Y}{2}$$

Exercise 9.2. *(Bonus 5 points) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two σ-algebra independent of each other. Let X be an integrable random variable such that $\sigma(\sigma(X), \mathcal{G}_1)$ is independent of $\mathcal{G}_2$. Show that*

$$E\left[X|\sigma(\mathcal{G}_1, \mathcal{G}_2))\right] = E\left[X|\mathcal{G}_1\right]$$

Exercise 9.3. *(5 points) Let X be a positive random variable. Show that*

$$E\left[X\right] = \int_0^\infty P\left[X > x\right] dx$$

Exercise 9.4. *(20 points)*

Let X and Y be two random variable with a joint distribution $P_{(X,Y)}$ absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Show that

- *show that $P_{(X,Y)}$ has a density, that it, there exists a measurable function $f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}_+$, such that*

  $$E\left[g(X,Y)\right] = \int_{\mathbb{R}^2} g(x,y) f_{(X,Y)}(x,y) dx dy$$

  for every positive measurable function $g : \mathbb{R}^2 \to [0, \infty[$.

- *show that P_X as well as P_Y are also absolutely continuous with respect to Lebegue. Provide an expression for their density f_X and f_Y respectively.*

- *show that $P_{(X,Y)} = P_X \otimes K$ for some stochastic kernel*

  $$K(x, B) = \int_B f_{(Y|X)}(x,y) dy$$

  where $f_{(Y|X)} : \mathbb{R}^2 \to \mathbb{R}$ is to be determined as a function of f_X, f_Y and $f_{(X,Y)}$.

- *show that*

  $$E\left[g(X,Y)|X\right] = \int_{\mathbb{R}^2} f_{Y|X}(X,y) dy$$

  for every positive measurable random variable $g : \mathbb{R}^2 \to \mathbb{R}$.

- *Suppose that $f_{(X,Y)} = 21_{[0,1]}(x+y)1_{[0\infty[}(x)1_{[0,\infty[}(y)$. Compute $E[\exp(Y)|X]$ explicitly.*

Exercise 9.5. *(5 points) Let (X_n) be a sequence of independent random variables such that each X_n is $\mathcal{N}(\mu, \sigma)$-distributed with $\mu \in \mathbb{R}$ and $\sigma > 0$, that is, they are all identically distributed with a probability density function with respect to Lebesgue given by*

$$f_{X_n}(x) = f_{X_1}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

We define the process M as

$$M_0 = 1, \quad \textit{and} \quad M_t = \exp\left(\sum_{s=1}^{t} X_s - \frac{n\sigma^2}{2}\right)$$

Show that M is a converging martingale and compute its limit.

Due date: Upload before Monday 2015.11.30 14:00.