# "STOCHASTIC PROCESSES" – HOMEWORK SHEET 13

Given a continuous square integrable martingale $M$ and a process $H \in \mathcal{L}^2(P \otimes d\langle M\rangle)$ we denote the stochastic integral of $H$ with respect to $M$ either

$$H \bullet M \quad \text{or} \quad \int H dM \quad \text{or} \quad \int H_s dM_s$$

**Exercise 13.1.** *Let $M \in \mathcal{M}_c^2$, $\alpha \in \mathbb{R}$ and $G, H \in \mathcal{L}^2(P \otimes d\langle M\rangle)$. Show that*

- $(\alpha H + G) \bullet M = \alpha H \bullet M + G \bullet M$;
- $(1_{[0,\tau]} H) \bullet M = H \bullet M^\tau = (H \bullet M)^\tau$.

For $M, N \in \mathcal{M}_c^2$, we define the co-variations $\langle M, N\rangle$ of $M$ and $N$ by means of the polar formula

$$\langle M, N\rangle = \frac{\langle M + N\rangle - \langle M - N\rangle}{4}$$

which is a continuous process of bounded variations – not necessarily increasing.

**Exercise 13.2.** *Show for $G, H \in \mathcal{S}$ and $M, N \in \mathcal{M}_c^2$ that it holds*

$$\langle \int G dM\rangle = \int G_s^2 d\langle M\rangle_s$$

$$\langle \int G dM, \int H dN\rangle = \int G_s d\langle M, \int H dN\rangle_s$$

$$= \int G_s H_s d\langle M, N\rangle_s$$

*Always with $G, H \in \mathcal{S}$, and the Doob-Meyer decomposition, follows the chain rule*

$$\int G\,(H \bullet M) = \int G H dM$$

**Exercise 13.3.** *Let $M \in \mathcal{M}_c^2$. We know from the Doob-Meyer decomposition that*

$$M^2 = \tilde{M} + \langle M\rangle$$

*for some martingale $\tilde{M}$. Show using Ito's formula that*

$$\tilde{M} = 2 \int M_s dM_s$$

**Exercise 13.4.** *Let $B$ be a Brownian motion.*

- *Show using Ito's Formula that the process $X$ given by*

$$X_t = e^{t/2}\cos(B_t)$$

*is a martingale.*

- *Consider the following stochastic differential equation*

$$dX_t = X_t\left(\mu_t dt + \sigma_t dB_t\right), \quad X_0 = 1$$

*where $\mu_t$ and $\sigma_t$ are uniformly bounded progressive processes and $\sigma_t > \varepsilon > 0$ for every $t$. Show using Ito's formula that it has a solution given by*

$$X_t = \exp\left(\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dB_s\right)$$

- *Using the previous point show that*

$$X_t = \exp\left(-\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dB_s\right)$$

*is a martingale.*

- *Consider the following stochastic differential equation*

$$dX_t = -\theta X_t dt + \sigma dB_t, \quad X_0 = x$$

*for $\theta, \sigma > 0$. This stochastic differential equation describes exponential convergence to $0$. Show that it has a solution by considering the process $U_t = e^{\theta t} X_t$.*

**Exercise 13.5.** *Let $B$ be a Brownian motion on a filtrated probability space and for a fixed time horizon $T \in \mathbb{R}_+$, let $f : [0,T] \to [0,T]$ be a measurable function – note that $f$ is deterministic, that is do not depend on the state $\omega \in \Omega$ – such that*

$$E\left[\int_0^T f_t^2 dt\right] < \infty.$$

*For $0 \leq a \leq b \leq T$, define*

$$J_{a,b} = \int_a^b f_s dB_s.$$

*a) Show that $J_{a,b}$ is well defined, and normally distributed $\mathcal{N}(\mu, \sigma)$ where*[1]

$$\mu = 0 \quad \text{and} \quad \sigma^2 = \int_a^b f_s^2 ds$$

[1] The case where $\int_a^b f_s^2 ds = 0$ being a trivial case where the distribution has 0 variance and point mass 1 at 0.

b) *Show further that for $0 \le a \le b \le c \le d \le T$, the random vector $(J_{a,b}, J_{c,d})$ is a Gaussian vector, that is, every linear combination*

$$\alpha J_{a,b} + \beta J_{c,d}$$

*is normally distributed for every choice of $\alpha, \beta \in \mathbb{R}$ – the case where $\alpha = \beta = 0$ being the previous trivial case with point mass 1 at 0. Show finally that $J_{a,b}$ is independent of $J_{c,d}$.*

Hint: *Show first the case where $f$ is piecewise constant. Then approximate $f$ by a sequence $(f^n)$ of piecewise constant functions and shows that it converges to the good distribution.*

**Exercise 13.6 (Difficult).** *A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order $\alpha$ at $x \in D$ if there exists $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \le C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \le \delta$. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order $\alpha$, if it is locally Hölder continuous of order $\alpha$ at each $x \in D$.*

a) *Let $Z \sim \mathcal{N}(0, 1)$. Prove that $P[|Z| \le \varepsilon] \le \varepsilon$ for any $\varepsilon \ge 0$.*

b) *Let $B$ be a Brownian motion (without the assumption that it has continuous paths). Prove that for any $\alpha > 1/2$, $P$-almost all path of the Brownian motion $B$ are nowhere on $[0, 1]$ locally Hölder-continuous of order $\alpha$.*

   Hint: *Take any $M \in \mathbb{N}$ satisfying $M(\alpha - 1/2) > 1$ and show that the set*

$$\{\omega \colon t \mapsto B_t(\omega) \text{ is locally Hölder continuous at some } t \in [0, 1]\}$$

   *is contained in the set*

$$\bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \ge m} \bigcap_{k=0,\dots,n-M} \bigcap_{j=1,\dots,M} \left\{ \left| B_{\frac{k+j}{n}} - B_{\frac{k+j-1}{n}} \right| \le C \frac{1}{n^\alpha} \right\}$$

c) *The Kolmogorov-Čentsov theorem states that any process $X$ on $[0, T]$ satisfying*

$$E\left[|X_t - X_s|^\gamma\right] \le C\,|t - s|^{1+\beta}, \quad 0 \le s, t \le T$$

   *where $\gamma, \beta, C > 0$, has a version which is locally Hölder-continuous of order $\alpha$ for all $\alpha < \beta/\gamma$. Use this to deduce that Brownian motion has for every $\alpha < 1/2$ a version which is locally Hölder-continuous of order $\alpha$.*

**Due date:** Optional