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Shanghai Jiao Tong University
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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 1

Exercise 1.1. *(12 Points)*

(a) Let Ω be a state space, and let $(\mathcal{F}_i)$ be an arbitrary family of σ-algebra on Ω. Show that

$$\mathcal{F} = \bigcap \mathcal{F}_i = \{A \subseteq \Omega \colon A \in \mathcal{F}_i \text{ for all } i\},$$

is a σ-algebra. Conclude that for a collection $\mathcal{C}$ of subsets of Ω,

$$\sigma(\mathcal{C}) := \bigcap \{\mathcal{F} \colon \mathcal{F} \text{ is a } \sigma\text{-algebra with } \mathcal{C} \subseteq \mathcal{F}\},$$

is the unique smallest σ-algebra containing $\mathcal{C}$.

(b) Give an example where the union of two σ-algebras is not a σ-algebra.

(c) Let $\Omega = \mathbb{R}$, and $\mathcal{F} = \mathcal{B}(\mathbb{R})$ the Borel σ-algebra of $\mathbb{R}$, that is, the σ-algebra generated by the collection $\{O \colon O \text{ open set in } \mathbb{R}\}$. It holds that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_i)$ for each $i = 0, \ldots, 17$ where

$$\begin{aligned}
\mathcal{C}_0 &= \{O \colon O \text{ open subset of } \mathbb{R}\} & \mathcal{C}_1 &= \{F \colon F \text{ closed subset of } \mathbb{R}\}\\
\mathcal{C}_2 &= \{]a,b[\colon a \leq b \text{ with } a,b \in \mathbb{R}\} & \mathcal{C}_3 &= \{[a,b] \colon a \leq b \text{ with } a,b \in \mathbb{R}\}\\
\mathcal{C}_4 &= \{]a,b] \colon a \leq b \text{ with } a,b \in \mathbb{R}\} & \mathcal{C}_5 &= \{[a,b[\colon a \leq b \text{ with } a,b \in \mathbb{R}\}\\
\mathcal{C}_6 &= \{]-\infty,b] \colon b \in \mathbb{R}\} & \mathcal{C}_7 &= \{]-\infty,b[\colon b \in \mathbb{R}\}\\
\mathcal{C}_8 &= \{[a,\infty[\colon a \in \mathbb{R}\} & \mathcal{C}_9 &= \{]a,\infty[\colon a \in \mathbb{R}\}\\
\mathcal{C}_{10} &= \{]a,b[\colon a \leq b \text{ with } a,b \in \mathbb{Q}\} & \mathcal{C}_{11} &= \{[a,b] \colon a \leq b \text{ with } a,b \in \mathbb{Q}\}\\
\mathcal{C}_{12} &= \{]a,b] \colon a \leq b \text{ with } a,b \in \mathbb{Q}\} & \mathcal{C}_{13} &= \{[a,b[\colon a \leq b \text{ with } a,b \in \mathbb{Q}\}\\
\mathcal{C}_{14} &= \{]-\infty,b] \colon b \in \mathbb{Q}\} & \mathcal{C}_{15} &= \{]-\infty,b[\colon b \in \mathbb{Q}\}\\
\mathcal{C}_{16} &= \{[a,\infty[\colon a \in \mathbb{Q}\} & \mathcal{C}_{17} &= \{]a,\infty[\colon a \in \mathbb{Q}\}
\end{aligned}$$

By definition $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_0)$. Show the assertion for the cases $i = 1, 9$ and 12.

Exercise 1.2. *(12 Points)*

(a) Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ be two measurable spaces. Given a function $X : \Omega \to S$, show that the collection of sets

$$\sigma(X) := \left\{X^{-1}(B) = \{\omega \in \Omega \colon X(\omega) \in B\} \colon B \in \mathcal{S}\right\},$$

is a σ-algebra on Ω.

Give a simple example where

$$\{X(A) = \{X(\omega) \colon \omega \in A\} \colon A \in \mathcal{F}\}$$

is not a σ-algebra on S.

(b) Let $(\Omega, \mathcal{F})$, $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be three measurable spaces. Given a $\mathcal{F}$-$\mathcal{S}$-measurable function $X : \Omega \to S$ and a $\mathcal{S}$-$\mathcal{T}$-measurable function $Y : S \to T$, show that $Z = Y \circ X : \Omega \to T$ is a $\mathcal{S}$-$\mathcal{T}$-measurable function.

(c) Let $(\Omega, \mathcal{F})$ be a measurable space, and X, Y be random variables as well as (X_n) be a sequence of random variables. Show that

- *$aX + bY$ is a random variable for every $a, b \in \mathbb{R}$;*
- *XY is a random variable;*
- $\max(X, Y)$ *and* $\min(X, Y)$ *are random variables;*
- $\sup X_n$ *and* $\inf X_n$ *are extended real valued random variables;*[1]
- $\liminf X_n := \inf_n \sup_{k \geq n} X_k$ *and* $\limsup X_n := \inf_n \sup_{k \geq n} X_k$ *are extended real valued random variables;*
- *$A := \{\lim X_n$ exists$\} := \{\omega\colon \lim X_n(\omega)$ exists$\} = \{\liminf X_n = \limsup X_n\}$ is measurable.*

Exercise 1.3. *(12 Points)*
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}$ be a random variable. Define

$$F(t) := P[X \leq t], \quad t \in \mathbb{R}.$$

which is called the cumulative distribution function of X. *Show that*

1) *$F : \mathbb{R} \to \mathbb{R}$ is increasing, $\lim_{t \to -\infty} F(t) = 0$, $\lim_{t \to \infty} F(t) = 1$ and F is right-continuous.*[2]

2) *F is measurable for the Borel σ-algebra;*

3) *F has at most countably many discontinuous points.*

Exercise 1.4. *(Bonus 12 Points)*
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(A_i)_{i \in I}$ be an arbitrary family – not necessarily countable – of measurable sets. Suppose that $P[A_i \cap A_j] = 0$ for every $i, j \in I$ with $i \neq j$ and $P[A_i] > 0$ for every $i \in I$. Show that the family $(A_i)_{i \in I}$ is at most countable.

Due date: Upload before Monday 2015.09.28 14:00.

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[1] With respect to the Borel σ-algebra on $[-\infty.\infty]$ generated by the metric $d(x, y) = |\arctan(x) - \arctan(y)|$ that coincide with the euclidean topology on $\mathbb{R}$. An extended random variable $X : \Omega \to [-\infty, \infty]$ is measurable if and only if $\{X \leq t\} \in \mathcal{F}$ for every $t \in \mathbb{R}$.

[2] That is $\lim_{s \nearrow t} F(s) = F(t)$.