"STOCHASTIC PROCESSES" – HOMEWORK SHEET 5

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtrated probability space with $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots}$.

Exercise 5.1. *(10 points)*
Provide an example of a martingale $X = (X_t)$ such that $\sup_t E[|X_t|] < \infty$ and $X_t \to X_\infty$ P-almost surely for some X_∞ but for which however it does not hold $E[|X_t - X_\infty|] \to 0$.

Exercise 5.2. *(10 points)*
Let X be an adapted and integrable stochastic process. Show that if $E[X_\tau] = E[X_0]$ for every finite stopping time[1] τ then X is a martingale.

Exercise 5.3. *(20 points)*
Under the same assumptions as Exercise 4.3 of the Homework sheet 4, that is, let X be an adapted process such that $E[\sup_t |X_t|] < \infty$. For the questions a) to d), we assume that $\mathbf{T} = \{0, \ldots, T\}$ for a given time horizon $T \in \mathbb{N}$. Denote by $\mathcal{T}$ the set of all stopping times with values in $\mathbf{T}$. We define recursively

$$S_T = X_T \quad \text{and} \quad S_t = \max\left\{E\left[S_{t+1} | \mathcal{F}_t\right]; X_t\right\}, \quad t \leq T - 1.$$

and denote by $S = M - A$ the Doob decomposition of S where M is a martingale and A is a predictable and integrable process with $A_0 = 0$. We define

$$\tau_0 = \inf\left\{t \colon X_t = S_t\right\} \quad \text{and} \quad \tau_1 = \inf\left\{t \colon A_{t+1} > 0\right\} \wedge T$$

a) Show that τ_1 is a stopping time such that $\tau_0 \leq \tau_1$;

b) Show that for every $\sigma \in \mathcal{T}$ the following assertions are equivalent:
- *(i) $E[X_\sigma] = \sup_{\tau \in \mathcal{T}} E[X_\tau]$;*
- *(ii) $X_\sigma = S_\sigma$ and S^σ is a martingale;*
- *(iii) $\tau_0 \leq \sigma \leq \tau_1$ and $E[X_\sigma] = E[S_\sigma]$.*

c) Let $\mathcal{M}_0$ be the set of all martingale $Y = (Y_t)$ such that $Y_0 = 0$. Show that

$$\max_{\tau \in \mathcal{T}} E\left[X_\tau\right] = \min_{Y \in \mathcal{M}_0} E\left[\max_{0 \leq t \leq T} \left(X_t - Y_t\right)\right]$$

d) Suppose that $T = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{F}_2 = 2^\Omega$. The process X is given by

$$X_0 \equiv 10, \quad X_1(\omega) = \begin{cases} 5 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 30 & \text{otherwise} \end{cases}, \quad X_3(\omega) = \begin{cases} 5 & \text{if } \omega = \omega_1 \\ 15 & \text{if } \omega = \omega_2 \\ 10 & \text{if } \omega = \omega_3 \\ 30 & \text{if } \omega = \omega_4 \end{cases}$$

compute τ_0 and τ_1.

[1] That is $P[\tau \leq T] = 1$ for some $T \in \mathbb{N}$

e) From now on, we consider that $\mathbf{T} = \mathbb{N}_0$, $T = \infty$ and consider the process S defined in homework sheet 4.3 d). For $n \in \mathbb{N}$ we define $\mathcal{T}^n = \{\tau \in \mathcal{T} \colon n \leq \tau < \infty\}$ and $\tau_n = \inf\{t \colon n \leq t$ and $S_t = X_t\}$. Show that if $\tau_n < \infty$, then

$$E\left[X_{\tau_n}\right] = E\left[S_n\right] = \max_{\tau \in \mathcal{T}^n} E\left[X_\tau\right]$$

Due date: Upload before Monday 2015.11.02 14:00.