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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 6

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtrated probability space with $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\dots}$.

Exercise 5.1. *(10 points) Let X be a sub-martingale. Let σ and τ be two stopping times, such that $\sigma \leq \tau \leq T$ for an integer T. We set*

$$\tau_0 = 0$$

and recursively

$$\begin{aligned}
\tau_1(\omega) &= \inf\{t \colon \sigma(\omega) \leq t \leq \tau(\omega), t \geq \tau_0(\omega), X_t(\omega) < x\}\\
\tau_2(\omega) &= \inf\{t \colon \sigma(\omega) \leq t \leq \tau(\omega), t \geq \tau_1(\omega), X_t(\omega) > y\}\\
&\vdots\\
\tau_{2k-1}(\omega) &= \inf\{t \colon \sigma(\omega) \leq t \leq \tau(\omega), t \geq \tau_{2k-2}(\omega), X_t(\omega) < x\}\\
\tau_{2k}(\omega) &= \inf\{t \colon \sigma(\omega) \leq t \leq \tau(\omega), t \geq \tau_{2k-1}(\omega), X_t(\omega) > y\}
\end{aligned}$$

with the convention that the infimum over the empty set is infinite. Define

$$U_{[\![\sigma(\omega),\tau(\omega)]\!]}(x, y, X(\omega)) = \sup\{k \colon \tau_{2k}(\omega) < \infty\}$$

where $[\![\sigma(\omega), \tau(\omega)]\!] = \{t \in \mathbb{N} \colon \sigma(\omega) \leq t \leq \tau(\omega)\}$.

(i) Show that

$$(y - x)\, E\left[U_{[\![\sigma,\tau]\!]}(x, y, X)|\mathcal{F}_\sigma\right] \leq E\left[(X_\tau - x)^+\,|\mathcal{F}_\sigma\right] - E\left[(X_\sigma - x)^+\,|\mathcal{F}_\sigma\right]$$

(ii) Show that

$$P\left[\sup_t X_t \geq \lambda\right] \leq \frac{1}{\lambda} \sup_t E\left[X_t^+\right]$$

Exercise 5.2. *(20 points)*

We consider the following random walk starting at 0

$$S_0 = 0 \quad \textit{and} \quad S_t = \sum_{s=1}^{t} X_s, \quad t \geq 1$$

where

$$X_t(\omega) = \begin{cases} 1 & \textit{if } \omega_t = 1, \\ -1 & \textit{if } \omega_t = -1. \end{cases} \quad t \geq 1 \textit{ and } \omega = (\omega_t)_{t\in\mathbb{N}} \in \Omega = \{-1, 1\}^{\mathbb{N}}.$$

As for the filtration we take

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \textit{and} \quad \mathcal{F}_t = \sigma(X_s \colon 1 \leq s \leq t), \quad t \geq 1$$

On $\mathcal{F} = \otimes_{t\in\mathbb{N}}\{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\}$ and the unique probability on $\mathcal{F}$ which on the finite cylinder is given by

$$P[A] = p^l q^{t-l}$$

where $p \in [0,1]$, $q = 1-p$ and

$$A = \{\omega \in \Omega \colon \omega_{t_k} = e_k, k = 1, \dots n\}$$

for the finite number of times $t_1, \dots, t_n$, numbers $e_k \in \{-1, 1\}$ and l is equal to the numbers of those $e_k = 1$. In other words, this is the probability that we have head at time t_k when $e_k = 1$ and tail when $e_k = -1$ for the finite numbers of time $t_1, \dots, t_n$ but with a biased coin which provides a probability p of having head and $q = 1-p$ of having tail.

(i) Show that S is a martingale if and only if $p = 1/2$. Show in that case that

$$P\left[\liminf X_t = -\infty \text{ and } \limsup X_t = \infty\right] = 1$$

(ii) Show in the general case

$$P\left[\lim X_t = \infty\right] = 1, \quad \text{if } p > 1/2;$$

and

$$P\left[\lim X_t = -\infty\right] = 1, \quad \text{if } p < 1/2.$$

To do so, show first that $E[X_t | \mathcal{F}_{t-1}] = E[X_t]$ for every $t \geq 1$. Then, that the Doob-Meyer decomposition

$$S = M - A,$$

is such that $A_t = -t(2p-1)$ and M satisfies

$$P\left[\liminf M_t = -\infty \text{ and } \limsup M_t = \infty\right] = 1$$

and use exercise 3.1.

(iii) Suppose that $p = 1/2$. Define $\tau = \inf\{t \colon S_t = a \text{ or } S_t = -b\}$ for two integers a, b.

- *Show that $P[\tau < \infty] = 1$.*
- *Show that*

$$P\left[S_\tau = a\right] = \frac{b}{a+b}$$

  This is the probability that S reach the value a before hitting $-b$.
- *Show that $M = S_t^2 - t$ is a martingale. Deduce that*

$$E\left[\tau\right] = ab$$

(iv) Suppose that $p \neq 1/2$, in the general case, show that $M_t := (q/p)^{S_t}$ is a martingale. Define $\tau = \inf\{t \colon S_t = a \text{ or } S_t = -b\}$ for two integers a, b. Show that

$$P[S_\tau = a] = \frac{(q/p)^b - 1}{(q/p)^{a+b} - 1}$$

Due date: Upload before Monday 2015.11.02 14:00.