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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 7

Exercise 7.1. *(5 points)*

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X = (X_t)_{t\in\mathbb{N}}$ be a process of integrable independent random variables, that is

$$P\left[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\right] = \prod_{k=1}^{n} P[X_{t_k} \in A_k]$$

for every finite subset of times $t_1, \ldots t_n$ and Borel sets $A_1, \ldots A_n$. Suppose that $E[X_t] = 1$ for every t.

For the filtration

$$\mathcal{F}_t = \sigma(X_s \colon s \leq t),$$

Show that the process Y given by

$$Y_t = \prod_{s=0}^{t} X_s, \quad s \geq 0$$

is a martingale.

Exercise 7.2. *(10 points)*

In this exercise, we will consider a simple discrete price process model with finite horizon $T \in \mathbb{N}$.

We denote by $S = (S_t)_{0\leq t\leq T}$ the evolution of a stock price for the times $t = 0, \ldots, T$. Usually, stock prices are strictly positives and characterized by their returns

$$R_t = \frac{S_t - S_{t-1}}{S_t}, \quad 1 \leq t \leq T$$

which is the proportional gain/loss of the price evolution between $t-1$ and t. Or in other terms, if the returns $R = (R_t)_{1\leq t\leq T}$ are given, the stock price is then

$$S_t = S_0 \prod_{s=1}^{t} \left(1 + R_s\right), \quad 0 \leq t \leq T$$

for a given start price $S_0 > 0$. To guarantee that the stock price remains strictly positive, we assume that $R_t > -1$ for every $t = 1, \ldots T$.

Our simple model is as follows. Let

- $\Omega = \{-1, 1\}^T = \{\omega = (\omega_t)_{1\leq t\leq T} \colon \omega_t \in \{-1, 1\}\}$;
- *For $-1 < d < u$, (where d stands for down and u for up) we define*

$$R_t(\omega) := \begin{cases} u & \text{if } \omega_t = 1 \\ d & \text{if } \omega_t = -1 \end{cases}, \quad 1 \leq t \leq T$$

- *As for the filtration we take*

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_t = \sigma(R_s \colon 1 \leq s \leq t), \quad 1 \leq t \leq T$$

Show that if $-1 < d < 0 < u$, then there exists a unique probability measure P on 2^Ω such that S is a martingale. Hereby, show that this probability measure is such that $P[\omega_t = 1] = -d/(u-d)$. Show that under this measure, the random variables $R_1, \ldots, R_T$ are independent.

Exercise 7.3. *(Bonus 10 points) Find a probability space, $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}}$, a process X and a random variable X_T such that*

- *X is a martingale;*
- $\sup_t E[|X_t|] < \infty$*;*
- *$X_t \to X_T$ P-almost surely;*
- *but $E[|X_t - X|] \not\to 0$.*

Due date: Upload before Monday 2015.11.09 14:00.