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"STOCHASTIC PROCESSES" – HOMEWORK SHEET 10

Definition 10.1. Given a transition probability p on the countable state space S. A distribution π on S is called *stationary* if

- $\pi_x < \infty$ for every state x;
- $\pi_x = \sum_{y \in S} \pi_x p_{xy}$, for every state x.

If $\sum \pi_x = 1$, we say that π is a stationary probability distribution.

Exercise 10.2. *(10 points)*

• *Let X be a time homogeneous Markov chain with start distribution μ and transition probability matrix p. Suppose that μ is a stationary probability distribution, shows that*

$$P_\mu \left[X_t = x\right] = \mu_x$$

for every time t and state x.

• *Let $(Y_n)_{n \geq 1}$ and X_0 be independent and identically distributed random variables on $\mathbb{Z}^d$. Defining the random walk*

$$X_t = X_0 + \sum_{s \leq t} Y_s$$

we saw in the lecture that it is a time homogeneous Markov chain. Show that $\pi_x = 1$ for every $x \in \mathbb{Z}^d$ is a stationary distribution.

Exercise 10.3. *(10 points)*

Let X be a time homogeneous Markov chain with values on the state space $S = \{x_0, x_1, \dots, x_6\}$ and with transition probability matrix

$$P = \begin{bmatrix} 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 \end{bmatrix}.$$

- *As in the lecture, provide a graph that shows how the process evolves between the seven states.*
- *Give the recurrent and transient states.*

Exercise 10.4. *(10 points)*

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a probability space and X be an adapted process with value in $(S, \mathcal{S})$ whereby S is countable and $\mathcal{S} = 2^S$. Let further

$$\mathcal{G}_t = \sigma(X_s \colon s \geq t)$$

the backward filtration generated by X. Show that the following three assertions are equivalent

- *X is a Markov-Chain*[1]
- *For all t and bounded $\mathcal{G}_t$-measurable random variable Y holds*

$$E\left[Y|\mathcal{F}_t\right] = E\left[Y|X_t\right].$$

- *For all t, $B \in \mathcal{F}_t$ and $C \in \mathcal{G}_t$ it holds*

$$P\left[B \cap C|X_t\right] = P\left[B|X_t\right] P\left[C|X_t\right]$$

Exercise 10.5. *(Bonus 10 points)*
Let $S := \{1,2,3,4\}$ and consider a time homogeneous Markov Chain X on the canonical space with transition probability matrix

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a) As in the lecture, provide a graph that shows how the process evolves between the four states. Hereby, you can visualize which sequences are possible or not.

b) Compute the probability $P_x[X_t = 4$, for some time $t]$ where x is a state. That is the probability that X reaches at some moment the state 4 starting from x.

c) Let $\tau_{1,4} = \inf\{t\colon X_t = 1$ or $X_t = 4\}$ be the first "visiting" time of the Markov Chain of the states 1 or 4. Compute

$$E_{P_x}\left[\tau_{1,4}\right]$$

for every state x.

d) Compute the stationary distribution of this Markov Chain – inspire yourself from Exercise 10.2.

Due date: Upload before Monday 2015.12.07 14:00.

[1]Not necessarily time-homogeneous.