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WS 2015/2016

"STOCHASTIC PROCESSES" – HOMEWORK SHEET 11

Exercise 11.1. *Let $T_1, T_2, \ldots$ be a sequence of independent random variable all exponentially distributed with parameter $\lambda > 0$, that is,*

$$dP_{T_n} = \lambda e^{-\lambda t} dt, \quad \textit{for every } n$$

We define the discrete process S as

$$S_0 \quad \textit{and} \quad S_n = \sum_{k=1}^{n} T_k$$

which somehow model the number of persons arriving into a queue. We finally define the continuous time process

$$N_t = \max\left\{n \in \mathbb{N} \colon S_n \leq t\right\}, \quad 0 \leq t < \infty$$

representing the number of persons in the queue at time t and define

$$\mathcal{F}_t = \sigma\left(N_s \colon s \leq t\right)$$

Show that

(i) Show that for $0 \leq s \leq t$, it holds[1]

$$P\left[S_{N_s+1} > t | \mathcal{F}_s\right] = e^{-\lambda(t-s)}$$

(ii) (Difficult, bonus) Show that for $s \leq t$, $N_t - N_s$ is a Poisson distributed random variable with parameter $\lambda(t-s)$ independent of $\mathcal{F}_s$[2] *that is*

$$E\left[1_A P\left[N_t - N_s \leq k | \mathcal{F}_s\right]\right] = P\left[A\right] \sum_{j=0}^{k} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^j}{j!}$$

for every $A \in \mathcal{F}_s$.

(iii) Show that the compensated Poisson process

$$M_t := N_t - \lambda t, \quad 0 \leq t < \infty$$

is a Martingale.

[1]Hint: Show that for every $A \in \mathcal{F}_s$ and every n, there exists $\tilde{A} \in \sigma(T_1, \ldots, T_n)$ such that $A \cap \{N_s = n\} = \tilde{A} \cap \{N_s = n\}$ and use the independence of T_{n+1} from $(S_n, 1_{\tilde{A}})$ to show that

$$E\left[1_{A \cap \{N_s = n\}} P\left[S_{n+1} > t | \mathcal{F}_s\right]\right] = e^{-\lambda(t-s)} P\left[A \cap \{N_s = n\}\right].$$

[2]Hint: You can use the previous result to show that for every $A \in \mathcal{F}_s$ and n, it holds

$$E\left[1_{A \cap \{N_s = n\}} P\left[N_t - N_s \leq k | \mathcal{F}_s\right]\right] = P\left[A \cap \{N_s = n\}\right] \sum_{j=0}^{k} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^j}{j!}$$

(iv) Show that for any $c > 0$, it holds

$$\limsup_{t\to\infty} P\left[\sup_{s\leq t} M_s \geq c\sqrt{\lambda t}\right] \leq \frac{1}{c\sqrt{2\pi}} \tag{11.1}$$

$$\liminf_{t\to\infty} P\left[\inf_{s\leq t} M_s \leq -c\sqrt{\lambda t}\right] \leq \frac{1}{c\sqrt{2\pi}} \tag{11.2}$$

$$E\left[\sup_{s\leq u\leq t}\left(\frac{M_u}{u}\right)^2\right] \leq \frac{4t\lambda}{s^2} \tag{11.3}$$

the latter inequality being for every $0 < s < t$.[3]

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t<\infty}, P)$ be a filtrated probability space.

Definition 11.2. A stochastic process B is called a Brownian Motion if

(I) B is adapted;

(II) $B_0 = 0$ almost surely;

(III) B has continuous path almost surely.[4]

(IV) $B_t - B_s$ is independent of $\mathcal{F}_s$ and $B_t - B_s \sim \mathcal{N}(0, t-s)$.

Exercise 11.3. *Let B be a Brownian motion. Show that*

(i) B is a martingale;

(ii) $B_t^2 - t$ is a martingale[5]

(iii) $\exp(\sigma B_t - \sigma^2 t/2)$ is a martingale for every $\sigma > 0$.

(iv) $1/\sigma^2 B_{\sigma t}$ is a Brownian motion with respect to the filtration $(\mathcal{F}_{\sigma t})_{0\leq t<\infty}$ for all $\sigma > 0$;

(v) For fixed s, $B_{t+s} - B_s$ is a Brownian motion with respect to $(\mathcal{F}_{t+s})_{0\leq t<\infty}$.

(vi) Use the last two point to show that the Brownian motion is non-differentiable at any t almost surely.

Due date: Upload before Monday 2015.12.14 14:00.

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[3] Recall Stirling's asymptotic behavior $n! \sim \sqrt{2\pi n}(n/e)^n$.

[4] That is $P[\{\omega\colon t \mapsto B_t(\omega) \text{ is continuous}\}] = 1$.

[5] You may use without proof that $B_t - B_s$ has the same distribution as B_{t-s} for every $0 \leq s \leq t$.