Teacher: Samuel Drapeau
Teaching Assistant: Zhang Yaoyuan

Shanghai Jiao Tong University
WS 2015/2016

"STOCHASTIC PROCESSES" – HOMEWORK SHEET 12

Exercise 12.1 (Easy). *1) Let $X = (X_t)_{0 \leq t \leq T}$ be a martingale on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. Show that if $X_t \geq 0$ P-almost surely, then holds for P-almost all $\omega \in \Omega$:*

$$X_t(\omega) = 0 \text{ for some } t \quad \text{implies} \quad X_s(\omega) = 0 \text{ for every } s = t+1, \ldots, T$$

2) Let $Y_1, \ldots, Y_t$ be independent random variables such that $Y_t \sim \mathcal{N}(0,1)$ on some probability space $(\Omega, \mathcal{F}, P)$. Consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma(Y_1, \ldots, Y_t)$. We consider the process

$$S_0 > 0, \quad S_t = S_0 \exp\left(\sum_{s=1}^{t} (\sigma_s Y_s + \mu_s)\right)$$

where σ_t, μ_t are constant for $t = 1, \ldots, T$ such that $\sigma_t \neq 0$. Let further

$$S_t^0 = (1+r)^t$$

for some constant $r > -1$. For which values of σ_t is the price process

$$X_t = \frac{S_t}{S_t^0}$$

a martingale.

Exercise 12.2 (Insider Problem). *Let $Y_1, \ldots, Y_T$ be independent identically distributed random variables such that $E[Y_t] = 0$ for every t and not identically constant on some probability space $(\Omega, \mathcal{F}, P)$. We consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma(Y_1, \ldots, Y_t)$ and process*

$$X_0 := 1, \quad X_t := X_0 + \sum_{s=1}^{t} Y_s.$$

We interpret this process as a stock price which is fair in the sense that it is a martingale and therefore does not brings any gain or loss in expectation. And for every strategy H, that is predictable process, the investment gain $H \bullet X_T$ at time T does not brings in average more than $H_0 X_0$ due to Doob's optional sampling theorem.

We extend the filtration with the information provided by X_T, that is

$$\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t, X_T), \quad t = 0, \ldots, T$$

This can be interpreted as the information of an insider knowing for whatever reason the terminal value of the price at time T. We denote the non-insider filtration $\mathbb{F}$ and the insider filtration $\tilde{\mathbb{F}}$. Show that

(i) X is a martingale under the filtration $\mathbb{F}$. Show that X can not be a martingale under the insider filtration $\tilde{\mathbb{F}}$. However, the process

$$\tilde{X}_t = X_t - \sum_{s=0}^{t-1} \frac{X_T - X_s}{T - s}, \quad t = 0, \ldots, T$$

is a martingale under $\tilde{\mathbb{F}}$.

(ii) With the information about the terminal value X_T of the stock, it is possible to realize arbitrage gains. It means that you can find a predictable process but with respect to $\tilde{\mathbb{F}}$ such that starting with 0 money, that is $H_0 = 0$, you end up with positive gains and even strict gains with strict positive probability. That is

$$P\left[H \bullet X_T \geq 0\right] = 1 \quad \textit{and} \quad P\left[H \bullet X_T > 0\right] > 0$$

Find the best "insider strategy" – that is $\tilde{\mathbb{F}}$-predictable process H with $H_0 = 0$ – that brings the maximum of gains among the insider strategies such that $|H_s| \leq 1$ for every $s = 1, \ldots, T$.

Exercise 12.3. *Let $f : [0, \infty[\to \mathbb{R}$ be a function. We define*

- *the variations of f as the function*

$$S_t = \sup_{\Pi=\{0=t_0\leq t_1\leq\cdots\leq t_n=t\}} \sum_{k=1}^{n} \left|f_{t_k} - f_{t_{k-1}}\right|, \quad t \in [0, \infty[$$

- *the quadratic variations of f as*

$$\langle f\rangle_t = \sup_{\Pi=\{0=t_0\leq t_1\leq\cdots\leq t_n=t\}} \sum_{k=1}^{n} \left|f_{t_k} - f_{t_{k-1}}\right|^2, \quad t \in [0, \infty[$$

We say that f has

- *bounded variations if $S_t < \infty$ for every t;*
- *quadratic variations if $\langle f\rangle_t < \infty$ for every t.*

(i) show that if f has bounded variations, then

$$S_t - f_t \quad \textit{and} \quad S_t + f_t$$

are both increasing functions.

(ii) Show that if $t \mapsto f_t$ is Lipschitz continuous, then f has bounded variations;

(iii) Show that if $t \mapsto f_t$ has bounded variations, then $\langle f\rangle_t = 0$ for every t.

Exercise 12.4. *Let B be the Brownian Motion (as in the previous exercise sheet) and consider a fixed time horizon $T < \infty$. Recall that you showed that $\langle B\rangle_t = t$, hence $d\langle B\rangle_t = dt$. Hence we will consider the space $\mathcal{L}^2 := \mathcal{L}^2(P \otimes dT)$ of those processes $H = (H_t)_{0\leq t\leq T}$ which are progressive and such that*

$$E\left[\int_0^T H_t^2 d\langle B\rangle_t\right] = E\left[\int_0^T H_t^2 dt\right] < \infty.$$

For a fixed time horizon $T < \infty$, define the process

$$H_t^n = \sum_{k=1}^{n} B_{t_{k-1}^n} 1_{]t_{k-1}^n, t_k^n]}(t), \quad 0 \leq t \leq T$$

where $t_k^n = kT/n$, $k = 0, \ldots, n$.

(i) Though $B_{t_{k-1}^n}$ is $\mathcal{F}_{t_{k-1}^n}$-measurable, it is now uniformly bounded and therefore not element of $\mathcal{S}$ as given in the lecture. Show however that it belongs to $\mathcal{L}^2$.

(ii) Show that $H^n \to B$ in $\mathcal{L}^2$ – for the L^2-norm. In particular, $B \in \mathcal{L}^2$.

(iii) Show that there exists a random variable $I_T \in \mathcal{L}^2$ such that

$$(H^n \bullet B)_T = \sum_{k=1}^{n} H^n_{t^n_k} \left(B_{t^n_k} - B_{t^n_{k-1}}\right) = \sum_{k=1}^{n} B_{t^n_{k-1}} \left(B_{t^n_k} - B_{t^n_{k-1}}\right)$$

converges in $\mathcal{L}^2$ to I_T. We denote this random variable the stochastic integral of B, that is

$$I_T := \int_0^T B_t dB_t$$

(iv) Using the relation $b(a-b) = (a^2 - b^2 - (a-b)^2)/2$, show using the approximation above that

$$\int_0^T B_t dB_t = \frac{1}{2}\left(B_T^2 - T\right)$$

Due date: Upload before Monday 2015.12.23 14:00.